5 - Usual functions

1. The exponential function

Definition: For $x+iy \in \mathbb{C}$, $x,y \in \mathbb{R}$, we set $e^z := \underbrace{e^x}_{\text{real exponential}} \underbrace{e^{iy}}_{\cos y + i\sin y}$

it defines the (complex) exponential function $\exp : \begin{array}{l} \mathbb{C} \longrightarrow \mathbb{C} \\ z \longmapsto e^z \end{array}$

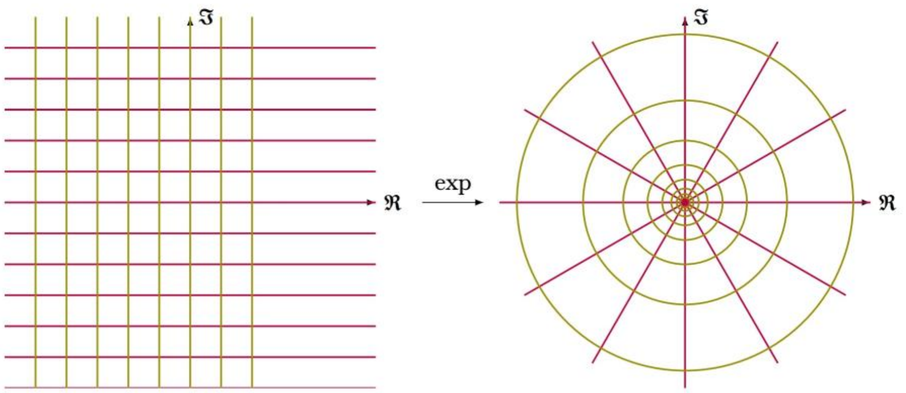
Prop:
- $\mathrm{Re}(e^z) = e^{\mathrm{Re}(z)} \cos(\mathrm{Im}(z))$ and $\mathrm{Im}(e^z) = e^{\mathrm{Re}(z)} \sin(\mathrm{Im}(z))$
- $|e^z| = e^{\mathrm{Re}(z)}$
- $\forall z_1, z_2 \in \mathbb{C}$, $e^{z_1+z_2} = e^{z_1} e^{z_2}$
- exp is $2\pi i$ periodic : $\forall z \in \mathbb{C}$, $e^{z+2i\pi} = e^z$
- $\mathrm{Range}(\exp) = \mathbb{C} \setminus \{0\}$, particularly $\forall z \in \mathbb{C}$, $e^z \neq 0$
- $e^z = 1 \iff \exists n \in \mathbb{Z}$, $z = 2i\pi n$

Δ
- $e^{x+iy} = e^x \cos y + i e^x \sin y$ so $\mathrm{Re}(e^{x+iy}) = e^x \cos y$ and $\mathrm{Im}(e^{x+iy}) = e^x \sin y$
- $|e^{x+iy}| = |e^x e^{iy}| = |e^x| \cdot |e^{iy}| = e^x (\cos^2 y + \sin^2 y)^{1/2} = e^x$
- $e^{(x_1+iy_1)+(x_2+iy_2)} = e^{(x_1+x_2)+i(y_1+y_2)} = e^{x_1+x_2} e^{i(y_1+y_2)} = \underbrace{e^{x_1} e^{x_2}}_{\text{real exp}} \underbrace{e^{iy_1} e^{iy_2}}_{\text{last week}} = e^{x_1} e^{iy_1} e^{x_2} e^{iy_2} = e^{x_1+iy_1} e^{x_2+iy_2}$
- $e^{2in\pi} = \cos(2n\pi) + i\sin(2n\pi) = 1$
  so $e^{z+2in\pi} = e^z e^{2in\pi} = e^z$
- Let $z = \rho e^{i\theta} \in \mathbb{C} \setminus \{0\}$, ie $\rho > 0$, $\theta \in \mathbb{R}$

  since $\exp : \mathbb{R} \longrightarrow \mathbb{R}_{>0}$ is surjective, $\exists r \in \mathbb{R}$ s.t. $\rho = e^r$

  then $z = e^{r+i\theta}$

  Moreover, if $z \in \mathbb{C}$, $|e^z| = e^{\mathrm{Re}(z)} \neq 0$ so $e^z \neq 0$
- $e^z = 1 \Rightarrow \begin{cases} |e^z| = 1 \\ e^x \cos y = 1 \\ e^x \sin y = 0 \end{cases} \Rightarrow \begin{cases} e^x = 1 \\ \cos y = 1 \\ \sin y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 2n\pi \text{ for some } n \in \mathbb{Z} \end{cases}$ $\square$



• Complex logarithm

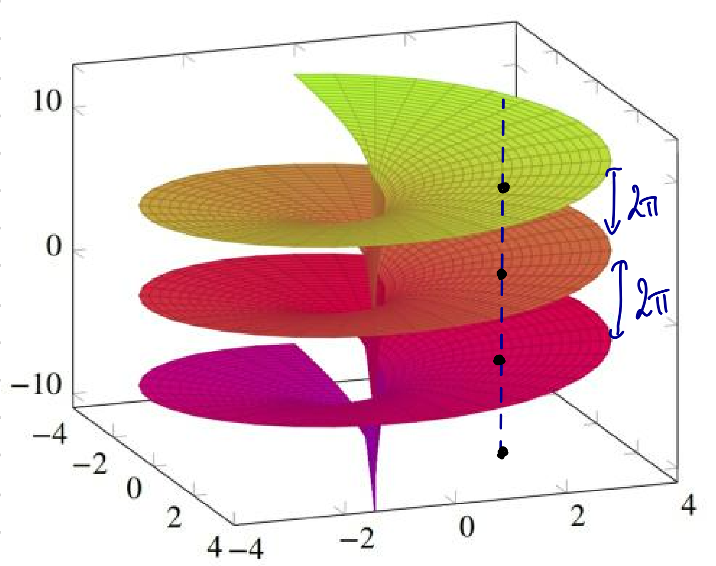
We want to define the complex logarithm as the inverse of the complex exponential as in the real case, but exp is not injective (it is $2\pi i$ periodic): let $z \in \mathbb{C}\setminus\{0\}$ then

$$e^{\omega} = z \iff \omega \equiv \log|z| + i\arg(z) \mod 2i\pi$$

Here "log" is the natural log ie of base e also denoted ln

Δ $$e^{a+ib} = z \iff \begin{cases} e^a = |z| \\ b = \arg z \end{cases} \iff \begin{cases} a = \log|z| \\ b = \arg z \end{cases} \quad \square$$

So we get a "multivalued function" $z \longmapsto \log|z| + i\arg(z)$ well defined only up to $2i\pi$: $\log(z) = \log|z| + i\,\mathrm{Arg}(z) + 2i\pi\mathbb{Z} = \{\log|z| + i\,\mathrm{Arg}(z) + 2i\pi m : m \in \mathbb{Z}\}$



← "Graph" of the imaginary part of the multivalued log

In order to remove the indeterminacy (ie to get a well defined function with values in $\mathbb{C}$), we need to shrink the domain of exp (to make it injective)

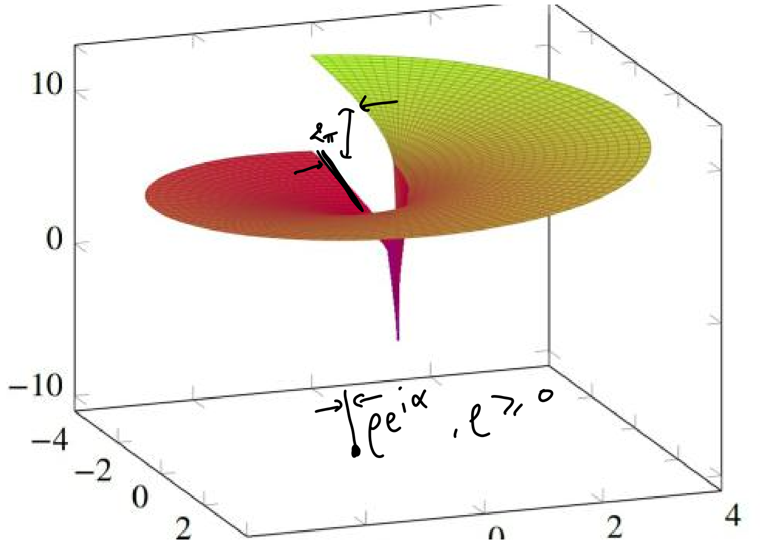
For $\alpha \in \mathbb{R}$, we set

$$\tilde{B}_\alpha := \{z \in \mathbb{C} : \alpha \leq \mathrm{Im}(z) < \alpha + 2\pi\}$$

then $\exp_{|\tilde{B}_\alpha} : \tilde{B}_\alpha \longrightarrow \mathbb{C}\setminus\{0\}$ is a bijection, but its inverse φ_α is not continuous:

for $\varepsilon > 0$ small: $\varphi_\alpha(e^{i(\alpha-\varepsilon)}) = \varphi_\alpha(e^{i(\alpha-\varepsilon+2\pi)}) = i(\alpha - \varepsilon + 2\pi)$ $\quad i(\alpha-\varepsilon+2\pi) \in \tilde{B}_\alpha$

so $\lim_{\varepsilon\to 0^+} \varphi_\alpha(e^{i(\alpha-\varepsilon)}) = i(\alpha+2\pi) \neq i\alpha = \varphi_\alpha(e^{i\alpha})$



discontinuity

To fix that we shrink again the domain (the first inequality is strict)

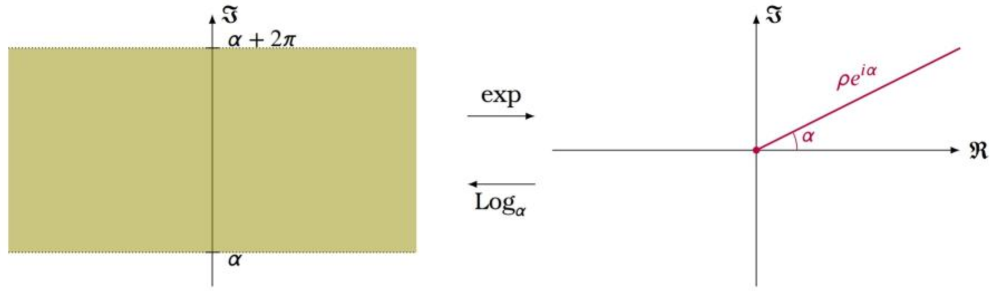
set $B_\alpha := \{ z \in \mathbb{C} : \alpha < \Im m(z) < \alpha + 2\pi \}$

then $\exp : B_\alpha \longrightarrow \mathbb{C} \setminus \underbrace{\{\rho e^{i\alpha}, \rho \geq 0\}}$ is a bijection

we remove a semiline corresponding to the image of $\Im m z = \alpha$

its inverse $\mathrm{Log}_\alpha : \mathbb{C} \setminus \{\rho e^{i\alpha}, \rho \geq 0\} \longrightarrow \{ z \in \mathbb{C} : \alpha < \Im m(z) < \alpha + 2\pi \}$

is continuous, it is the complex logarithm branch corresponding to the cut α



Note that : $\mathrm{Log}_\alpha : \mathbb{C} \setminus \{\rho e^{i\alpha}, \rho \geq 0\} \longrightarrow \{ z \in \mathbb{C} : \alpha < \Im m(z) < \alpha + 2\pi \}$

$z \longmapsto \log|z| + i \,\mathrm{Arg}_\alpha(z)$

where $\alpha < \mathrm{Arg}_\alpha(z) < \alpha + 2\pi$

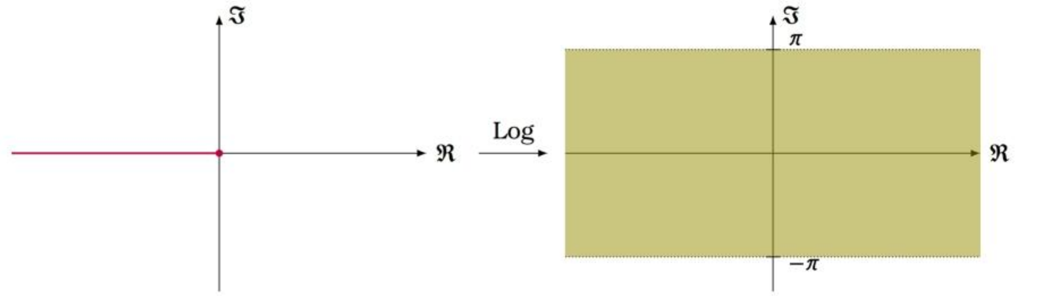
When α is not specified, it (often) means that $\alpha = -\pi$

$$\mathrm{Log}: \begin{array}{ccc} \mathbb{C}\setminus\mathbb{R}_{\leq 0} & \longrightarrow & \{z \in \mathbb{C} : -\pi < \mathfrak{Im}(z) < \pi\} \\ z & \longmapsto & \log(|z|) + i\,\mathrm{Arg}(z) \end{array}$$

where $\mathrm{Arg}(z) \in (-\pi, \pi)$

it is then called the principal branch of the complex logarithm

Comment ($\alpha = -\pi$): $\mathrm{Log}(x+iy) = \log\left(\sqrt{x^2+y^2}\right) + i \begin{cases} \arctan(y/x) & \text{if } x > 0 \\ \pi/2 - \arctan(x/y) & \text{if } y > 0 \\ -\pi/2 - \arctan(x/y) & \text{if } y < 0 \end{cases}$



The notation is really ambiguous and depends a lot on the authors so be careful!
In the textbook, "log" is multivalued / well defined modulo $2i\pi$
"Log" is the principal branch.

⚠ $\mathrm{Log}_\alpha(z_1 z_2) \neq \mathrm{Log}_\alpha(z_1) + \mathrm{Log}_\alpha(z_2)$
this property is no longer true

Eg: I assume that $\alpha = -\pi$ (principal branch)
Let $z = e^{2i\frac{\pi}{3}}$ then $\qquad \in B_{-\pi} = \{z \in \mathbb{C} : -\pi < \mathfrak{Im}(z) < \pi\}$

$$\mathrm{Log}(z^2) = \mathrm{Log}\left(e^{4i\pi/3}\right) = \mathrm{Log}\left(e^{-2i\pi/3}\right) = -2i\frac{\pi}{3} \neq 4i\frac{\pi}{3} = 2\,\mathrm{Log}(z)$$

($4i\pi/3 \notin B_{-\pi}$)

__However:__ $\log(z_1 z_2) \equiv \log(z_1) + \log(z_2) \mod 2i\pi$
$\log(z_1/z_2) \equiv \log(z_1) - \log(z_2) \mod 2i\pi$

• Power function

For $\omega \in \mathbb{C}$, and $z \in \mathbb{C}\setminus\{0\}$, we set $z^\omega := e^{\omega \log z} = \{e^{\omega \operatorname{Log} z} e^{2im\pi\omega} : m \in \mathbb{Z}\}$

⚠ Since $\log$ is only defined modulo $2i\pi$, z^ω is only defined up to a factor $e^{2im\pi\omega}$, $m \in \mathbb{Z}$ (it is, once again, multivalued)

Eg: $\sqrt{z} = z^{1/2} = \{e^{\frac{1}{2}\operatorname{Log} z} e^{im\pi} : m \in \mathbb{Z}\} = \{\pm e^{\frac{1}{2}\operatorname{Log} z}\}$

Indeed, the square root is well defined only up to a sign.

Nonetheless, if $\omega \in \mathbb{Z}$ then z^ω is well-defined: $e^{2i\pi m\omega} = 1$ since $m\omega \in \mathbb{Z}$.

and then $z^\omega = z \times \cdots \times z$ ω times if $\omega \geq 0$

or $z^\omega = \dfrac{1}{z \times \cdots \times z}$ if $\omega < 0$.

⚠ Be careful when writing $(z_1 z_2)^\omega = z_1^\omega z_2^\omega$

it is only true modulo a factor $e^{2i\pi\omega m}$, $m \in \mathbb{Z}$ (ie as sets)

⚠ $z^{\omega_1} z^{\omega_2} \neq z^{\omega_1+\omega_2}$ is generally false even seen as "multivalued function"

$z^{\omega_1} z^{\omega_2}$ is well defined up to a factor $e^{2i\pi(m\omega_1 + k\omega_2)}$, $m, k \in \mathbb{Z}$

$z^{\omega_1+\omega_2}$ is well defined up to a factor $e^{2i\pi m(\omega_1+\omega_2)}$, $m \in \mathbb{Z}$

If we see these multivalued functions as sets then $z^{\omega_1+\omega_2} \subset z^{\omega_1} z^{\omega_2}$

Trigonometric functions

Definitions:

$\cos : \mathbb{C} \longrightarrow \mathbb{C}$ $\qquad$ $\sin : \mathbb{C} \longrightarrow \mathbb{C}$

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

Proposition:

① $\forall z \in \mathbb{C}, \quad \cos^2 z + \sin^2 z = 1$

② $\forall z, w \in \mathbb{C}, \quad \sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$

③ $\forall z, w \in \mathbb{C}, \quad \cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$

④ $\forall z \in \mathbb{C}, \quad \sin(-z) = -\sin(z)$

⑤ $\forall z \in \mathbb{C}, \quad \cos(-z) = \cos(z)$

⑥ $\forall z \in \mathbb{C}, \sin(z + 2\pi) = \sin(z)$

⑦ $\forall z \in \mathbb{C}, \quad \cos(z + 2\pi) = \cos(z)$

Δ ① $\cos^2 z + \sin^2 z = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2$

$$= \frac{e^{2iz} + e^{-2iz} + 2 - e^{2iz} - e^{-2iz} + 2}{4} = 1$$

② $\sin(z)\cos(w) + \cos(z)\sin(w)$

$$= \frac{e^{iz} - e^{-iz}}{2i} \cdot \frac{e^{iw} + e^{-iw}}{2} + \frac{e^{iz} + e^{-iz}}{2} \cdot \frac{e^{iw} - e^{-iw}}{2i}$$

$$= \frac{e^{i(z+w)} + \cancel{e^{i(z-w)}} - \cancel{e^{i(-z+w)}} - e^{i(-z-w)} + e^{i(z+w)} - \cancel{e^{i(z-w)}} + \cancel{e^{i(-z+w)}} - e^{i(-z-w)}}{4i}$$

$$= \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i} = \sin(z+w)$$

④ $\sin(-z) = \dfrac{e^{-iz} - e^{iz}}{2i} = -\dfrac{e^{iz} - e^{-iz}}{2i} = -\sin(z)$

⑤ $\sin(z + 2\pi) = \dfrac{e^{i(z+2\pi)} - e^{-i(z+2\pi)}}{2i} = \dfrac{e^{iz}e^{2i\pi} - e^{-iz}e^{-2i\pi}}{2i} = \dfrac{e^{iz} - e^{-iz}}{2i} = \sin(z)$

$\square$

Prop: $\cos : \mathbb{C} \to \mathbb{C}$ are surjective
$\sin : \mathbb{C} \to \mathbb{C}$

Δ Let $w \in \mathbb{C}$. We want to find $z \in \mathbb{C}$ s.t. $\cos(z) = w$

$$\Leftrightarrow \frac{e^{iz}+e^{-iz}}{2} = w$$

$$\Leftrightarrow e^{iz}+e^{-iz} = 2w$$

$$\Leftrightarrow \left(e^{iz}\right)^2 - 2w\, e^{iz} + 1 = 0$$

$\exists v \in \mathbb{C}\setminus 0$, s.t. $v^2 - 2wv + 1 = 0$

and since Range(exp) $= \mathbb{C}\setminus\{0\}$, $\exists z \in \mathbb{C}$, $v = e^{iz}$ $\square$

Hence it is ~~false~~ that $|\cos z| \leq 1$
false !!!

Study of cos • the horizontal line $\{z \in \mathbb{C} : \Im m(z) = c\}$ is mapped to

$$\{\cos(x+ic) : x \in \mathbb{R}\} = \left\{\cos(x)\left(\frac{e^{c}+e^{-c}}{2}\right) + i\sin(x)\left(\frac{e^{-c}-e^{c}}{2}\right) : x \in \mathbb{R}\right\}$$

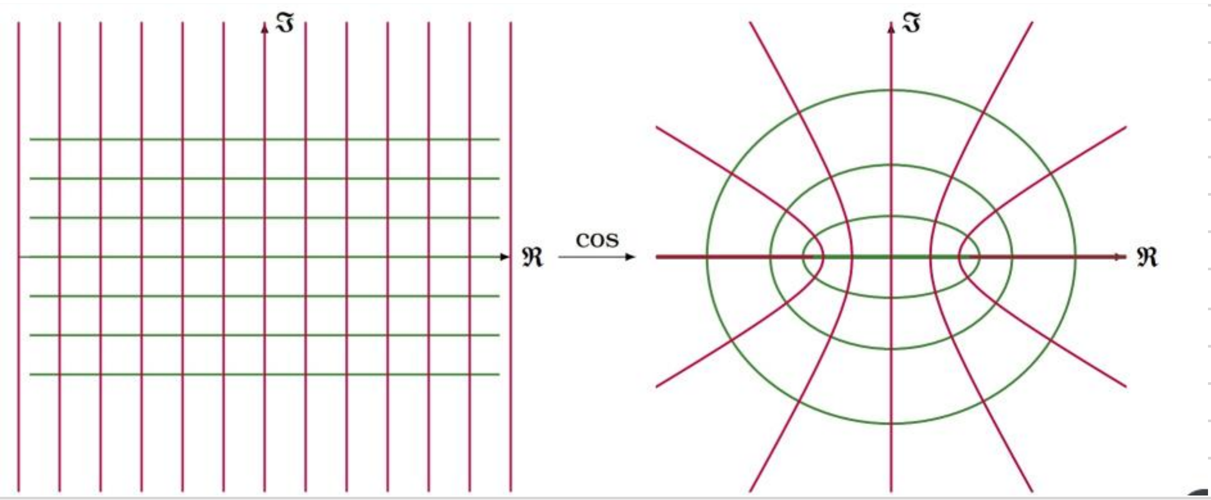
which is the ellipse with semi-axis $\frac{e^{c}+e^{-c}}{2}$ and $\left|\frac{e^{-c}-e^{c}}{2}\right|$

• the vertical line $\{z \in \mathbb{C} : \Re e(z) = c\}$ is mapped to

$$\{\cos(c+iy) : y \in \mathbb{R}\} = \{\cos(c)\cosh(y) - i\sin(c)\sinh(y) : y \in \mathbb{R}\}$$

which is a branch of the hyperbola with semiaxis $|\cos(c)|$, $|\sin(c)|$

focus is 1 for all these conics, so each hyperbola is perpendicular to each ellipse



Homework: study sin similarly

Proposition : $\cos z = 0 \Leftrightarrow \exists n \in \mathbb{Z},\ z = \frac{\pi}{2} + \pi n$

$\sin z = 0 \Leftrightarrow \exists n \in \mathbb{Z},\ z = \pi n$

$\triangleright$ $\cos z = 0 \Leftrightarrow e^{iz} + e^{-iz} = 0$

$\Leftrightarrow e^{2iz} = -1$

$\Leftrightarrow \exists n \in \mathbb{Z},\ 2iz = \pi i + 2i\pi n$

$\sin z = 0 \Leftrightarrow e^{iz} - e^{-iz} = 0$

$\Leftrightarrow e^{2iz} = 1$

$\Leftrightarrow \exists n \in \mathbb{Z},\ 2iz = 2i\pi n$

$\square$

Definition : $\tan : \mathbb{C} \setminus \{\frac{\pi}{2} + \pi n,\ n \in \mathbb{Z}\} \longrightarrow \mathbb{C}$

$$\tan z = \frac{\sin z}{\cos z}$$

Definition : $\cot : \mathbb{C} \setminus \{\pi n : n \in \mathbb{Z}\} \longrightarrow \mathbb{C}$

$$\cot z = \frac{\cos z}{\sin z}$$

Inverse trigonometric functions

We want to find the inverse of cos (but cos is not injective so we will get a multivalued function)

$$\cos(w) = z \iff \frac{e^{iw} + e^{-iw}}{2} = z \iff \left(e^{iw}\right)^2 - 2ze^{iw} + 1 = 0$$

But: $v^2 - 2zv + 1 = 0 \iff v = z + \sqrt{z^2-1}$ — the square root is already multivalued: it is well-defined up to its sign only

hence $\cos(w) = z \iff e^{iw} = z + \sqrt{z^2-1}$

$$\iff iw = \log\left(z + \sqrt{z^2-1}\right)$$

$$\iff w = -i\log\left(z + \sqrt{z^2-1}\right)$$

square root is multivalued

log is multivalued

hence $\boxed{\arccos(z) = -i\log\left(z + \sqrt{z^2-1}\right)}$ (multivalued, the value is a set)

similarly, we get

$$\boxed{\begin{aligned} \arcsin(z) &= -i\log\left(iz + \sqrt{1-z^2}\right) \\ \arctan(z) &= \frac{i}{2}\log\left(\frac{1-iz}{1+iz}\right), \quad z \neq \pm i \end{aligned}}$$

particularly
Range $(\tan) = \mathbb{C} \setminus \{\pm i\}$

Homework: check the formulae for arcsin / arctan.

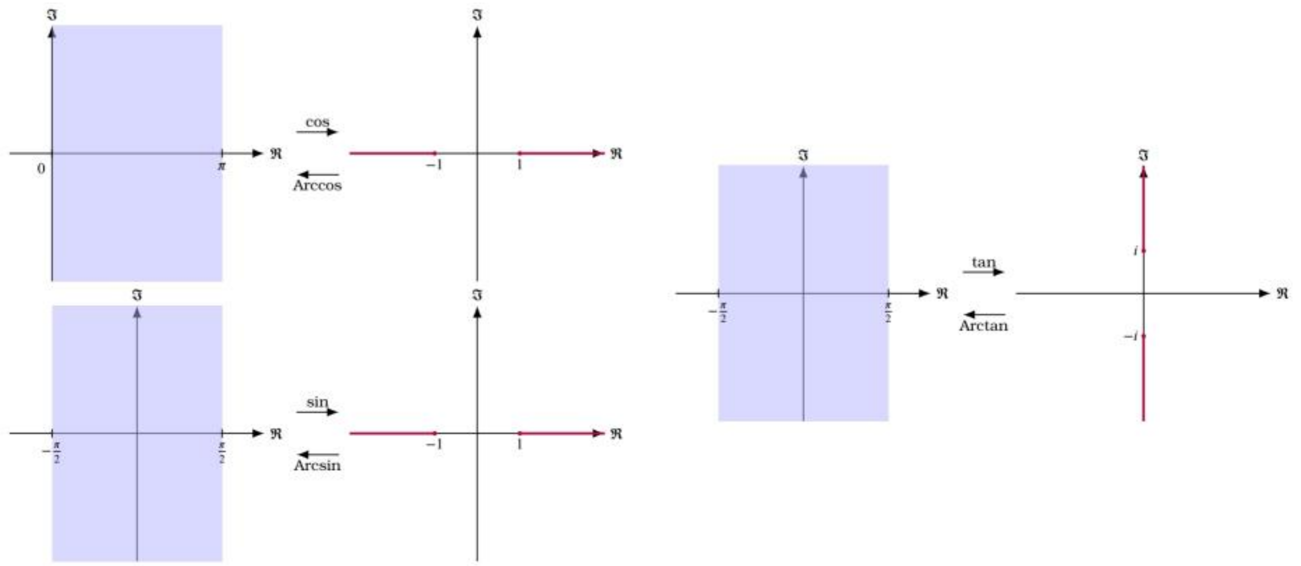
One may check that

$$\cos : \{z \in \mathbb{C} : 0 < \Re(z) < \pi\} \to \mathbb{C} \setminus ((-\infty,-1] \cup [1,+\infty))$$

$$\sin : \left\{z \in \mathbb{C} : -\frac{\pi}{2} < \Re(z) < \frac{\pi}{2}\right\} \to \mathbb{C} \setminus ((-\infty,-1] \cup [1,+\infty))$$

$$\tan : \left\{z \in \mathbb{C} : -\frac{\pi}{2} < \Re(z) < \frac{\pi}{2}\right\} \to \mathbb{C} \setminus ((-i\infty,-i] \cup [i,i\infty))$$

are bijective. (Homework)



Their inverses are taken as "principal branches" (well-defined functions)
We get the following formulae:

$$\text{Arccos} : \begin{array}{ccc} \mathbb{C} \setminus ((-\infty,-1] \cup [1,+\infty)) & \to & \{z \in \mathbb{C} \,:\, 0 < \Re(z) < \pi\} \\ z & \mapsto & -i \,\mathrm{Log}\left(z + \sqrt{z^2-1}\right) \end{array}$$

$$\text{Arcsin} : \begin{array}{ccc} \mathbb{C} \setminus ((-\infty,-1] \cup [1,+\infty)) & \to & \left\{z \in \mathbb{C} \,:\, -\frac{\pi}{2} < \Re(z) < \frac{\pi}{2}\right\} \\ z & \mapsto & -i \,\mathrm{Log}\left(iz + \sqrt{1-z^2}\right) \end{array}$$

$$\text{Arctan} : \begin{array}{ccc} \mathbb{C} \setminus ((-i\infty,-i] \cup [i,+i\infty)) & \to & \left\{z \in \mathbb{C} \,:\, -\frac{\pi}{2} < \Re(z) < \frac{\pi}{2}\right\} \\ z & \mapsto & \frac{i}{2} \,\mathrm{Log}\left(\frac{1-iz}{1+iz}\right) \end{array}$$

where: ① for $\sqrt{\cdot}$ we take the value whose real part is ≥ 0

② for log, we take its principal branch Log

Theorem: $\arccos(z) = \{ \pm \operatorname{Arccos}(z) + 2\pi n \ : \ n \in \mathbb{Z} \}$

$\arcsin(z) = \{ (-1)^n \operatorname{Arcsin}(z) + \pi n \ : \ n \in \mathbb{Z} \}$

$\arctan(z) = \{ \operatorname{Arctan}(z) + \pi n \ : \ n \in \mathbb{Z} \}$

Complex hyperbolic functions:

Definition: $\cosh : \mathbb{C} \longrightarrow \mathbb{C}$

$$\cosh(z) = \frac{e^{z}+e^{-z}}{2}$$

$\sinh : \mathbb{C} \longrightarrow \mathbb{C}$

$$\sinh(z) = \frac{e^{z}-e^{-z}}{2}$$

Proposition: $\cos(z) = \cosh(iz)$ and $\sin(z) = -i\sinh(iz)$

Δ $\cosh(iz) = \dfrac{e^{iz}+e^{-iz}}{2} = \cos z$ $\quad$ $-i\sin(iz) = -i\,\dfrac{e^{iz}-e^{-iz}}{2} = \dfrac{e^{iz}-e^{-iz}}{2i} = \sin(z)$ $\square$

Hence: you may derive the hyperbolic identities from the trigonometric ones

Homework: $\forall z \in \mathbb{C}$, $\cosh^2 z - \sinh^2 z = 1$

$\forall z, w \in \mathbb{C}$, $\cosh(z+w) = \cosh(z)\cosh(w) + \sinh(z)\sinh(w)$

$\forall z, w \in \mathbb{C}$, $\sinh(z+w) = \sinh(z)\cosh(w) + \cosh(z)\sinh(w)$

$\forall x, y \in \mathbb{R}$, $\cos(x+iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$

$\forall x, y \in \mathbb{R}$, $\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$

① $\cosh^2(z) - \sinh^2(z) = \left(\cos\left(\frac{z}{i}\right)\right)^2 + \left(\sin\left(\frac{z}{i}\right)\right)^2 = 1$

② $\cosh(z+w) = \cos\left(\frac{z}{i}+\frac{w}{i}\right) = \cos\left(\frac{z}{i}\right)\cos\left(\frac{w}{i}\right) - \sin\left(\frac{z}{i}\right)\sin\left(\frac{w}{i}\right)$

$= \cos\left(\frac{z}{i}\right)\cos\left(\frac{w}{i}\right) + \left(-i\sin\left(\frac{z}{i}\right)\right)\left(-i\sin\left(\frac{w}{i}\right)\right)$

$= \cosh z \cosh w + \sinh z \sinh w$

③ $\cos(x+iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy)$

$= \cos x \cosh(i^2 y) - \sin(x)\left(-i\sinh(i^2 y)\right)$

$= \cos x \cosh(-y) + i\sin(x)\sinh(-y)$

$= \cos(x)\cosh(-y) - i\sin(x)\sinh(y)$