# 3 - Topology of $\mathbb{C}$

It is going to be the exact same thing as in the multivariable calculus class.

<u>Definition</u>: The open disk of $\mathbb{C}$ centered at $z_0 \in \mathbb{C}$ and of radius $\varepsilon > 0$ is

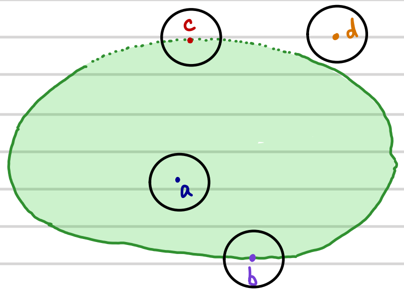
$$D_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$$

(You can also see the name $\varepsilon$-vicinity of $z_0$)

<u>Definition</u>: $V \subset \mathbb{C}$ is a neighborhood of $z_0$ if $\exists \varepsilon > 0$, $D_\varepsilon(z_0) \subset V$.

<u>Definition</u>: A subset $S \subset \mathbb{C}$ is bounded if there exists $r > 0$ s.t. $S \subset D_r(0)$

Let's fix a subset $S \subset \mathbb{C}$.




| | | | |
|---|---|---|---|
| $a \in S$ | $b \in S$ | $c \notin S$ | $d \notin S$ |
| $a \in \mathring{S}$ | $b \notin \mathring{S}$ | $c \notin \mathring{S}$ | $d \notin \mathring{S}$ |
| $a \in \overline{S}$ | $b \in \overline{S}$ | $c \in \overline{S}$ | $d \notin \overline{S}$ |
| $a \notin \partial S$ | $b \in \partial S$ | $c \in \partial S$ | $d \notin \partial S$ |

<u>Definition</u>: the interior of $S$ is $\mathring{S} := \{z \in \mathbb{C} : \exists \varepsilon > 0, D_\varepsilon(z) \subset S\}$
(or $S^{int}$)

<u>Definition</u>: the closure of $S$ is $\overline{S} := \{z \in \mathbb{C} : \forall \varepsilon > 0, D_\varepsilon(z) \cap S \neq \emptyset\}$

<u>Theorem</u>: $\mathring{S} \subset S \subset \overline{S}$

$\Delta$ $\mathring{S} \subset S$: let $z \in \mathring{S}$ then $\exists \varepsilon > 0$ s.t. $D_\varepsilon(z) \subset S$, hence $z \in D_\varepsilon(z) \subset S$, so $z \in S$

$S \subset \overline{S}$: let $z \in S$. Pick $\varepsilon > 0$, then $z \in S \cap D_\varepsilon(z)$

hence $\forall \varepsilon > 0$, $S \cap D_\varepsilon(z) \neq \emptyset$, and $z \in \overline{S}$ $\square$

<u>Definition</u>: the boundary of $S$ is $\partial S := \overline{S} \setminus \mathring{S}$

<u>Proposition</u>: $\overline{S} = S \cup \partial S$ and $\mathring{S} \cap \partial S = \emptyset$

$\Delta$ Trivial $\square$

<u>Theorem</u>: $\partial S = \{z \in \mathbb{C} : \forall \varepsilon > 0, D_\varepsilon(z) \cap S \neq \emptyset \text{ and } D_\varepsilon(z) \cap S^c \neq \emptyset\}$

$\Delta$ $x \in \overline{S} \Leftrightarrow \forall \varepsilon > 0$, $D_\varepsilon(z) \cap S \neq \emptyset$

$x \notin \mathring{S} \Leftrightarrow$ no $(\exists \varepsilon > 0, D_\varepsilon(z) \subset S) \Leftrightarrow \forall \varepsilon > 0, D_\varepsilon(z) \cap S^c \neq \emptyset$ $\square$

<u>Corollary</u>: $\partial S = \partial(S^c)$

$\Delta$ Note that $(S^c)^c = S$ $\square$

**Definition**: $S \subset \mathbb{C}$ is open if $\mathring{S} = S$

**Theorem**: $S$ open $\Leftrightarrow$ $S \cap \partial S = \emptyset$

$\Delta$ $\Rightarrow$: Assume that $S$ is open, then $S \cap \partial S = \mathring{S} \cap \partial S = \emptyset$

$\Leftarrow$: We prove the contrapositive: $S$ not open $\Rightarrow$ $S \cap \partial S \neq \emptyset$

Assume that $S$ is not open, then $\mathring{S} \subsetneq S$, ie $\exists x \in S \setminus \mathring{S}$

but $S \setminus \mathring{S} \subset \overline{S} \setminus \mathring{S} = \partial S$, so $x \in S \cap \partial S$ and $S \cap \partial S \neq \emptyset$ □

**Theorem**: $S$ open $\Leftrightarrow$ $\forall z \in S, \exists \varepsilon > 0, D_\varepsilon(z) \subset S$

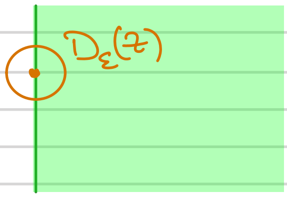
$\Delta$ $\Rightarrow$: let $z \in S$ then $z \in \mathring{S}$ since $\mathring{S} = S$, so $\exists \varepsilon > 0, D_\varepsilon(z) \subset S$

$\Leftarrow$: We already know that $\mathring{S} \subset S$, let's prove that $S \subset \mathring{S}$.

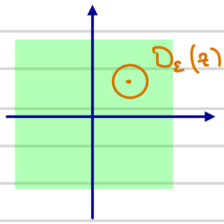
Let $z \in S$ then $\exists \varepsilon > 0, D_\varepsilon(z) \subset S$ hence $z \in \mathring{S}$

Therefore $S \subset \mathring{S}$. □

Eg: $\{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$ is not open




Eg: $\{z \in \mathbb{C} : \max(|\mathrm{Re}(z)|, |\mathrm{Im}(z)|) < 1\}$ is open




**Definition**: $S \subset \mathbb{C}$ is closed if $\overline{S} = S$

**Theorem**: $S$ closed $\Leftrightarrow$ $\partial S \subset S$

$\Delta$ $\Rightarrow$: $\partial S = \overline{S} \setminus \mathring{S} \subset \overline{S} = S$

$\Leftarrow$: $\overline{S} = S \cup \partial S = S$ □

**Theorem**: $S$ closed $\Leftrightarrow$ $S^c$ open

$\Delta$ $S$ closed $\Leftrightarrow \partial S \subset S \Leftrightarrow \partial(S^c) \subset S \Leftrightarrow \partial(S^c) \cap S^c = \emptyset \Leftrightarrow S^c$ open □

↳ $\partial S = \partial(S^c)$

Eg: $\{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$ is closed

$\{z \in \mathbb{C} : |z| \leq 1\}$ is closed

$D_\varepsilon(z)$ is not closed

⚠ $\{z \in \mathbb{C} : |z| < 1\}$ is open not closed

$\{z \in \mathbb{C} : |z| \leq 1\}$ is closed not open

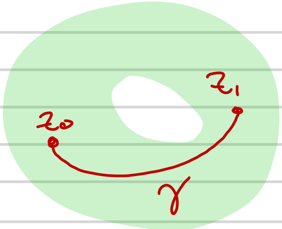
$\mathbb{C}$ is open and closed

$\{z \in \mathbb{C} : \mathrm{Re}(z) = 0, \mathrm{Im}(z) > 0\}$ is not open not closed

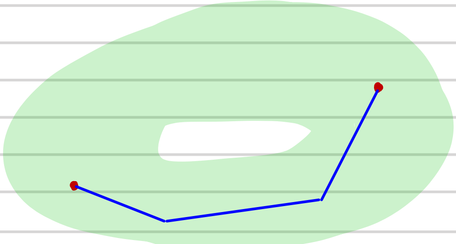
**Definition**: A subset $S \subset \mathbb{C}$ is path-connected if $\forall z_0, z_1 \in S$,
$\exists \gamma : [0,1] \longrightarrow \mathbb{C}$ continuous such that
- $\forall t \in [0,1]$, $\gamma(t) \in S$
- $\gamma(0) = z_0$
- $\gamma(1) = z_1$

meaning that both the real and imaginary parts of $\gamma$ are continuous

Eg:



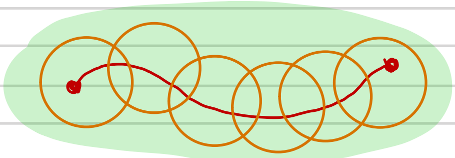

Eg



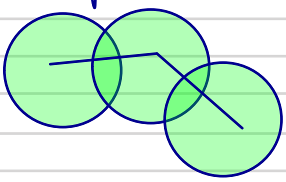
**Theorem**: An open subset $S \subset \mathbb{C}$ is path-connected if and only if for for $z, w \in S$, there exists a polygonal curve from $z$ to $w$ lying in $S$

ie: $\exists P_0, \dots, P_n$ s.t. the segment line $[P_i P_{i+1}] \subset S$ and $P_0 = z$, $P_n = w$

Δ Idea of proof: in this case, we may cover a path $\gamma$ from $z$ to $w$ with finitely many open disks lying in $S$

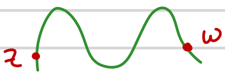


and we use the fact that the disks are convex:



□

**Remark**: the "open" assumption is very important here, for instance:
the curve



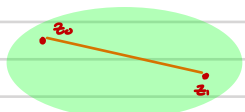

is path-connected, but there is no polygonal curve from $z$ to $w$ lying in it.

**Definition**: we say that an open set is connected if it is path-connected.

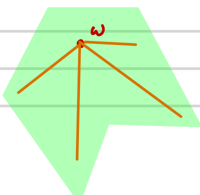
⚠ We only defined connectedness for open sets, there is a more general notion but connectedness and path-connectedness coincide for open sets

**Definition**: Domain := open and connected

**Definition**: $S \subset \mathbb{C}$ is convex if $\forall z_0, z_1 \in S$, $\forall t \in [0,1]$, $(1-t) z_0 + t z_1 \in S$




**Definition**: $S \subset \mathbb{C}$ is star-shaped if $\exists w \in S, \forall z \in S, \forall t \in [0,1]$, $(1-t) w + t z \in S$




$\emptyset$ is convex but not star-shaped

**Remark** convex + non-empty $\Rightarrow$ star-shaped $\Rightarrow$ path-connected

$\not\Leftarrow$ $\not\Leftarrow$

# The point at $\infty$

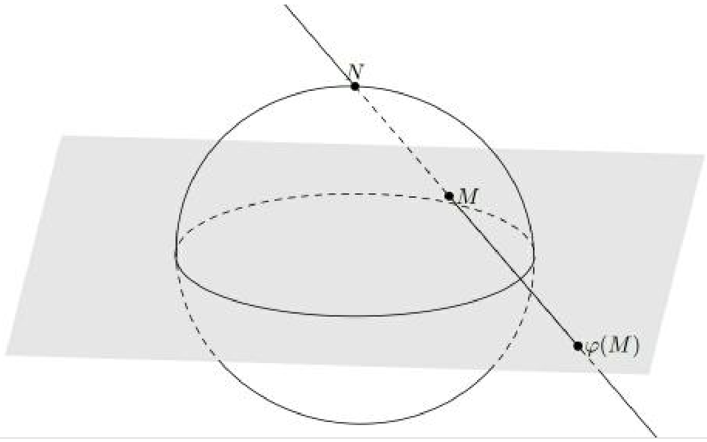
We are going to extend $\mathbb{C}$ by adding <u>one</u> point at $\infty$. There are different ways to do that, the model we are going to use is the Riemann Sphere.

$\mathbb{S}^2 := \{(r,s,t) \in \mathbb{R}^3 : r^2+s^2+t^2 = 1\}$, $N = (0,0,1)$ "the North Pole"

We identify $\mathbb{C}$ with the plane $P = \{t=0\} \subset \mathbb{R}^3$

We consider the stereographic projection with respect to $N$:

$$\varphi : \begin{array}{ccc} \mathbb{S}^2 \setminus \{N\} & \longrightarrow & P \\ M & \longmapsto & (NM) \cap P \end{array}$$ (mapping $M$ to the intersection of the line $(MN)$ with the plane $P$)




We may check that $\varphi$ is a bijection (see next page), then we may identify $\mathbb{C}$ with $\mathbb{S}^2 \setminus \{N\}$ and see $N$ as the point at $\infty$.

In practice we simply write $\hat{\mathbb{C}} = \mathbb{C} \sqcup \{\infty\}$.

Eg: the inversion extends to $\hat{\mathbb{C}}$: $\hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$

$$\begin{cases} z \longmapsto z^{-1} \text{ if } z \neq 0, \infty \\ \infty \longmapsto 0 \\ 0 \longmapsto \infty \end{cases}$$

<u>Definition</u>: We say that $V \subset \mathbb{C}$ is a neighborhood of the $\infty$ if its complement ${}^cV = \mathbb{C} \setminus V$ is bounded

<u>Theorem</u>: $V \subset \mathbb{C}$ is a neighborhood of $\infty$ iff $\exists R > 0$ s.t. $\{z \in \mathbb{C} : |z| > R\} \subset V$

$\Delta$ $V$ is a neighborhood of $\infty$ $\Leftrightarrow$ $\mathbb{C} \setminus V$ bounded

$\Leftrightarrow \exists R > 0,\ \mathbb{C} \setminus V \subset \{z \in \mathbb{C} : |z| \leq R\}$

$\Leftrightarrow \exists R > 0,\ \{z \in \mathbb{C} : |z| > R\} \subset V$ $\square$

We obtain a topology on $\hat{\mathbb{C}}$: $S \subset \hat{\mathbb{C}}$ is open if it is a neighborhood of each of its points

$S \subset \hat{\mathbb{C}}$ is open if • $S \subset \mathbb{C}$ is open

or • $S = \{\infty\} \cup K^c$ where $K \subset \mathbb{C}$ compact (closed and bounded)

This construction is a special case of a general topological construction called the "one point compactification" or "Alexandrov" compactification.

Proof that $\varphi$ is a bijection:

Take $M=(r,s,t) \in S^2 \setminus \{N\}$ and set $M' = \varphi(M)$

then $M' = (x,y,0)$ and $\overrightarrow{NM'} = \lambda \overrightarrow{NM}$

$$\text{ie} \quad \begin{pmatrix} x \\ y \\ -1 \end{pmatrix} = \lambda \begin{pmatrix} r \\ s \\ t-1 \end{pmatrix}$$

so $\lambda = \dfrac{1}{1-t}$, $x = \dfrac{r}{1-t}$, $y = \dfrac{s}{1-t}$

ie $\varphi(r,s,t) = \dfrac{r}{1-t} + i \dfrac{s}{1-t}$

We solve $\begin{cases} x = \dfrac{r}{1-t} \\ y = \dfrac{s}{1-t} \\ r^2+s^2+t^2=1 \end{cases}$ to find the inverse.

we get $[x(t-1)]^2 + [y(t-1)]^2 + t^2 = 1 \iff (x^2+y^2+1)t^2 - 2(x^2+y^2)t + (x^2+y^2-1) = 0$

so $t = \dfrac{x^2+y^2-1}{x^2+y^2+1}$ since $t \neq 1$

and $\varphi^{-1}(x+iy) = \left( \dfrac{2x}{x^2+y^2+1}, \dfrac{2y}{x^2+y^2+1}, \dfrac{x^2+y^2-1}{x^2+y^2+1} \right)$ $\square$

A comment about the line-circle equation:

$$a\bar{z}z - \bar{\gamma}z - \gamma\bar{z} + k = 0, \quad a,k \in \mathbb{R}, \quad \gamma \in \mathbb{C}, \quad |\gamma|^2 - ak > 0$$

If we set $\omega = z^{-1}$ (ie we swap $0$ and $\infty$ in $\hat{\mathbb{C}}$) then it becomes

$$a\frac{1}{\bar{\omega}}\frac{1}{\omega} - \bar{\gamma}\frac{1}{\omega} - \gamma\frac{1}{\bar{\omega}} + k = 0 \iff k\omega\bar{\omega} - \alpha\bar{\omega} - \bar{\alpha}\omega + a = 0$$

where $\alpha = \bar{\gamma}$

So the inversion maps "generalized line-circles" to "generalized line-circles"

and the equation makes sense in $\hat{\mathbb{C}}$.

⚠ Disclaimer: in $\hat{\mathbb{C}}$ we extended $i: z \longmapsto z^{-1}$ so that $\frac{1}{0} = \infty$

But be careful, $\infty$ is NOT the multiplicative inverse of $0$

$$0 \cdot \infty \neq 1$$

Actually we shouldn't say that "we can't divide by 0" but rather that we can't define a multiplicative inverse of 0.