University of Toronto – MAT334H1-F – LEC0101

Complex Variables

18 – The Riemann mapping theorem

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Theorem 1 (The Riemann mapping theorem)**.**
Let $U \subsetneq \mathbb{C}$ be a simply connected open subset which is not $\mathbb{C}$.
Then there exists a biholomorphism $f : U \to D_1(0)$ (i.e. f is holomorphic, bijective and f^{-1} is holomorphic).
*We say that U and $D_1(0)$ are **conformally equivalent**.*

Remark 2. Note that if $f : U \to V$ is bijective and holomorphic then f^{-1} is holomorphic too.
Indeed, we proved that if f is injective and holomorphic then f' never vanishes (Nov 30).
Then we can conclude using the inverse function theorem.

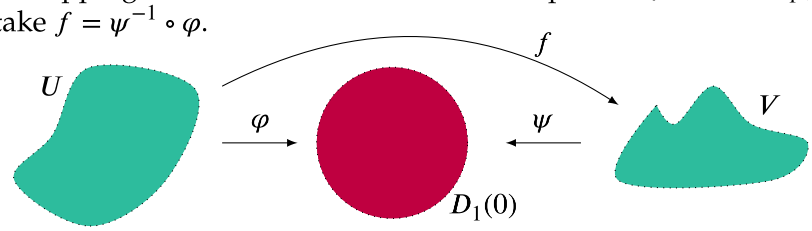
Note that this remark is false for $\mathbb{R}$-differentiability: define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$ then $f'(0) = 0$ and $f^{-1}(x) = \sqrt[3]{x}$ is not differentiable at 0.

Remark 3. The theorem is false if $U = \mathbb{C}$.
Indeed, by Liouville's theorem, if $f : \mathbb{C} \to D_1(0)$ is holomorphic then it is constant (as a bounded entire function), so it can't be bijective.

The Riemann mapping theorem states that up to biholomorphic transformations, the unit disk is a model for open simply connected sets which are not $\mathbb{C}$.
Otherwise stated, up to a biholomorphic transformation, there are only two open simply connected sets: $D_1(0)$ and $\mathbb{C}$. Formally:

Corollary 4. *Let $U, V \subsetneq \mathbb{C}$ be two simply connected open subsets, none of which is $\mathbb{C}$.*
Then there exists a biholomorphism $f : U \to V$ (i.e. f is holomorphic, bijective and f^{-1} is holomorphic).

Proof. By the Riemann mapping theorem, there exists biholomorphisms $\varphi : U \to D_1(0)$ and $\psi : V \to D_1(0)$.
Then we can simply take $f = \psi^{-1} \circ \varphi$.



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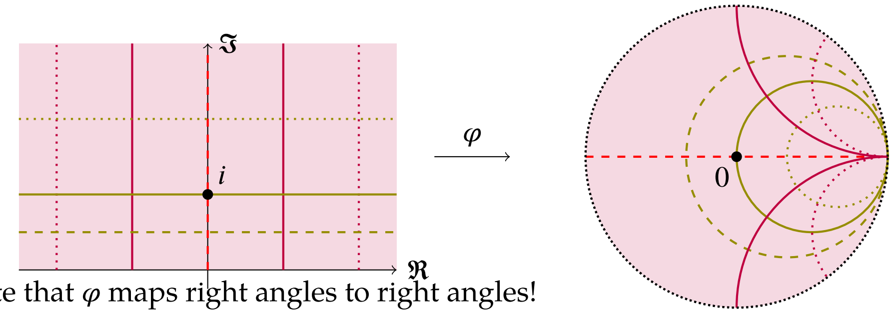
Corollary 5. *Let $U \subset \mathbb{C}$ be an open subset.*
Then U is simply connected if and only if it is homeomorphic to $D_1(0)$.

Proof.
$\Rightarrow$ Assume that $U \subsetneq \mathbb{C}$ is simply connected then there exists a biholomorphism $f : U \to D_1(0)$. Particularly f is a homeomorphism.
Note that $\mathbb{C}$ is also homeomorphic to $D_1(0)$.
$\Leftarrow$ Assume that there exists a homeomorphism $f : V \to U$ where $V = D_1(0)$.
Since V is simply connected, we get that U is too since simple connectedness is preserved by homeomorphisms. ■

Remark 6. Careful: the continuous image of a simply connected set may not be simply connected.
For instance $\exp(\mathbb{C}) = \mathbb{C} \setminus \{0\}$.

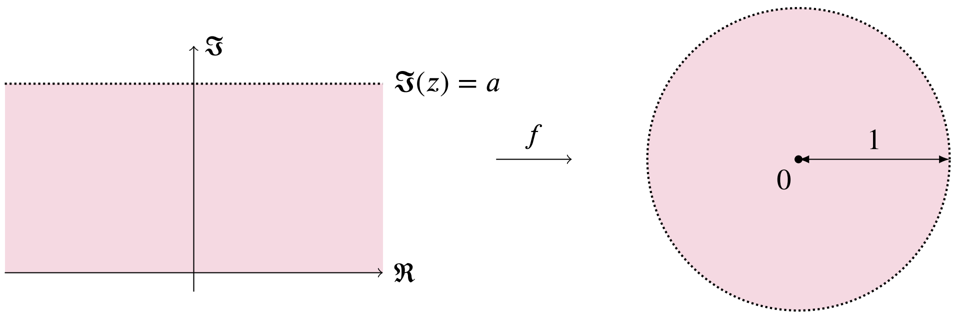
Example 7 (The Poincaré half-plane)**.**
We define the Poincaré half-plane by $\mathbb{H} = \{z \in \mathbb{C} \,:\, \Im(z) > 0\}$.
Then the mapping $\varphi : \mathbb{H} \to D_1(0)$ defined by $\varphi(z) = \dfrac{z-i}{z+i}$ is biholomorphic.
First check that φ is well-defined: $\forall z \in \mathbb{H}$, $z \neq -i$ and $\varphi(z) \in D_1(0)$.
Then note that φ is the restriction of a Möbius transformation $\hat{\varphi} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.
It is not too difficult to check that $\hat{\varphi}(\mathbb{R} \cup \{\infty\}) = S^1 (:= \{z \in \mathbb{C} \,:\, |z| = 1\})$.
The complement of $\mathbb{R} \cup \{\infty\}$ in $\widehat{\mathbb{C}}$ has two connected components which are $\mathbb{H}$ and $-\mathbb{H}$.
And $\widehat{\mathbb{C}} \setminus S^1$ has two connected components: $D_1(0)$ and $\{z \in \mathbb{C} \,:\, |z| > 1\} \cup \{\infty\}$.
Since $\varphi(i) = 0 \in D_1(0)$, we deduce that $\varphi(\mathbb{H}) = D_1(0)$.



Note that φ maps right angles to right angles!

Example 8 (A horizontal band)**.**
We set $\mathcal{B} := \{z \in \mathbb{C} \,:\, 0 < \Im(z) < a\}$, $a > 0$.
We know that $\psi : \mathcal{B} \to \mathbb{H}$ defined by $\psi(z) = e^{\frac{\pi}{a}z}$ is biholomorphic.
Hence $f = \varphi \circ \psi : \mathcal{B} \to D_1(0)$ is biholomorphic, where φ was defined in the previous example, i.e.

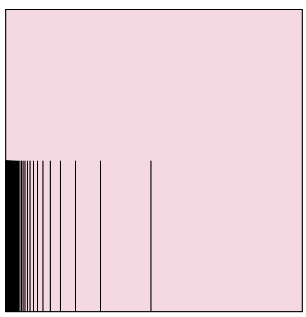
$$f(z) = \frac{e^{\frac{\pi}{a}z} - i}{e^{\frac{\pi}{a}z} + i}$$



Example 9. In practice the biholomorphism φ between U and $D_1(0)$ may be difficult to express explicitely. For instance, the following set is simply connected

$$U = \big((0,1) \times (0,1)\big) \setminus \left(\bigcup_{n \geq 2} \left\{ \frac{1}{n} \right\} \times \left(0, \frac{1}{2}\right) \right)$$

but the behavior of φ around the boundary of $D_1(0)$ is going to be quite complicated!



Or, even worse, take U to be the interior of the Koch snowflake.