

17 – Conformal mappings

Jean-Baptiste Campesato

November 30th, 2020 and December 2nd, 2020

1 Definition

Definition 1. Let $U \subset \mathbb{C}$ be open and $f : U \rightarrow \mathbb{C}$. Let $z_0 \in U$. Assume that there exists $r > 0$ such that $D_r(z_0) \subset U$ and $\forall z \in D_r(z_0) \setminus \{z_0\}$, $f(z) \neq f(z_0)$.

We say that f is **conformal at z_0** if f preserves angles at z_0 , i.e.

$$\lim_{\varepsilon \rightarrow 0^+} e^{-i\theta} \frac{f(z_0 + \varepsilon e^{i\theta}) - f(z_0)}{|f(z_0 + \varepsilon e^{i\theta}) - f(z_0)|}$$

exists and doesn't depend on $\theta \in \mathbb{R}$.

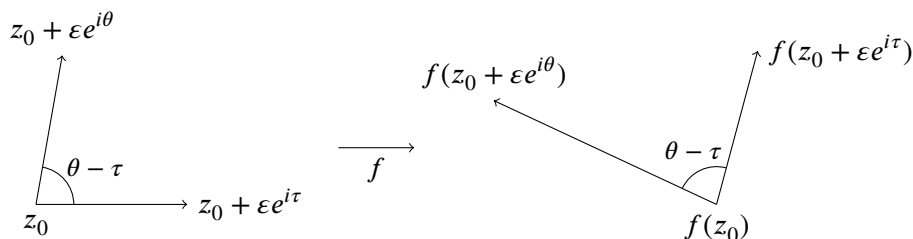
We say that f is conformal if it is conformal at every $z \in U$.

Remark 2. In the previous definition,

$$\frac{f(z_0 + \varepsilon e^{i\theta}) - f(z_0)}{|f(z_0 + \varepsilon e^{i\theta}) - f(z_0)|} \in S^1$$

is the direction from $f(z_0)$ to $f(z_0 + \varepsilon e^{i\theta})$.

Hence the definition means that the oriented angle of the images of two rays originated from z_0 is the same as the oriented angle of these two rays.



Example 3. Set $f(z) = az$ where $a \in \mathbb{C} \setminus \{0\}$. Then f is conformal at every $z_0 \in \mathbb{C}$, indeed

$$\lim_{\varepsilon \rightarrow 0^+} e^{-i\theta} \frac{f(z_0 + \varepsilon e^{i\theta}) - f(z_0)}{|f(z_0 + \varepsilon e^{i\theta}) - f(z_0)|} = \frac{a}{|a|}$$

exists and doesn't depend on $\theta \in \mathbb{R}$.

Example 4. Set $f(z) = z^2$. Then f is conformal at every $z_0 \in \mathbb{C} \setminus \{0\}$, indeed

$$\lim_{\varepsilon \rightarrow 0^+} e^{-i\theta} \frac{f(z_0 + \varepsilon e^{i\theta}) - f(z_0)}{|f(z_0 + \varepsilon e^{i\theta}) - f(z_0)|} = \frac{z_0}{|z_0|}$$

exists and doesn't depend on $\theta \in \mathbb{R}$.

However f is not conformal at 0 since $\lim_{\varepsilon \rightarrow 0^+} e^{-i\theta} \frac{f(\varepsilon e^{i\theta}) - f(0)}{|f(\varepsilon e^{i\theta}) - f(0)|} = e^{i\theta}$ depends on $\theta \in \mathbb{R}$.

2 Connection with holomorphicity/analyticity

Theorem 5. Let $U \subset \mathbb{C}$ be open and $f : U \rightarrow \mathbb{C}$.

1. If f is holomorphic/analytic at z_0 and if $f'(z_0) \neq 0$ then f is conformal at z_0 .
2. Conversely, if $\tilde{f} : \tilde{U} \rightarrow \mathbb{R}^2$ is \mathbb{R} -differentiable at (x_0, y_0) and $\text{Jac}_{\tilde{f}}(x_0, y_0) \neq \mathbf{0}$ and f is conformal at $z_0 = x_0 + iy_0$ then f is holomorphic at z_0 and $f'(z_0) \neq 0$.

Proof.

$$1. \lim_{\varepsilon \rightarrow 0^+} e^{-i\theta} \frac{f(z_0 + \varepsilon e^{i\theta}) - f(z_0)}{|f(z_0 + \varepsilon e^{i\theta}) - f(z_0)|} = \lim_{\varepsilon \rightarrow 0^+} \frac{f(z_0 + \varepsilon e^{i\theta}) - f(z_0)}{\varepsilon e^{i\theta}} \frac{|\varepsilon e^{i\theta}|}{|f(z_0 + \varepsilon e^{i\theta}) - f(z_0)|} = \frac{f'(z_0)}{|f'(z_0)|} \text{ exists and doesn't depend on } \theta \in \mathbb{R}.$$

2. Since $\text{Jac}_{\tilde{f}}(x_0, y_0) \neq \mathbf{0}$ we know that $f(z_0 + h) = f(z_0) + \alpha h + \beta \bar{h} + o(|h|)$ where $(\alpha, \beta) \neq (0, 0)$.

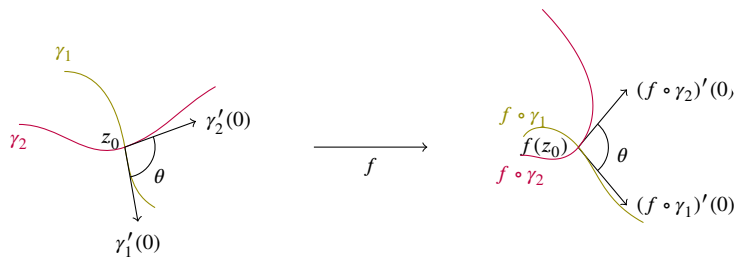
Since $\lim_{\varepsilon \rightarrow 0^+} e^{-i\theta} \frac{f(z_0 + \varepsilon e^{i\theta}) - f(z_0)}{|f(z_0 + \varepsilon e^{i\theta}) - f(z_0)|} = \frac{\alpha + \beta e^{-2i\theta}}{|\alpha + \beta e^{-2i\theta}|}$ doesn't depend on $\theta \in \mathbb{R}$, we obtain that $\beta = 0$ so that $f(z_0 + h) = f(z_0) + \alpha h + o(|h|)$, i.e. f is holomorphic at z_0 and $f'(z_0) = \alpha \neq 0$. ■

Corollary 6. Let $U \subset \mathbb{C}$ be open and $z_0 \in U$. Let $f : U \rightarrow \mathbb{C}$ be holomorphic/analytic. Then f is conformal at z_0 if and only if $f'(z_0) \neq 0$.

Remark 7. Here is another geometric interpretation of conformality for f holomorphic at z_0 with $f'(z_0) \neq 0$. Let $\gamma_1, \gamma_2 : (-1, 1) \rightarrow \mathbb{C}$ be two smooth curves such that $z_0 = \gamma_1(0) = \gamma_2(0)$ and $\gamma_i'(0) \neq 0$. Then f being conformal at z_0 means that for any such curves, the oriented angles $(\gamma_1'(0), \gamma_2'(0))$ and $((f \circ \gamma_1)'(0), (f \circ \gamma_2)'(0))$ coincide,

$$\text{i.e. } \arg(\gamma_2'(0)) - \arg(\gamma_1'(0)) \equiv \arg((f \circ \gamma_2)'(0)) - \arg((f \circ \gamma_1)'(0)) \pmod{2\pi}$$

Indeed, by the chain rule $\arg((f \circ \gamma_i)'(0)) \equiv \arg(f'(z_0)) + \arg(\gamma_i'(0)) \pmod{2\pi}$ if $f'(z_0) \neq 0$.



3 A global result

Theorem 8. Let $U \subset \mathbb{C}$ be open. Let $f : U \rightarrow \mathbb{C}$ be holomorphic/analytic. If f is injective on U then $\forall z \in U, f'(z) \neq 0$.

Proof. We are going to prove the contrapositive.

Assume that there exists $z_0 \in U$ such that $f'(z_0) = 0$.

Then in a neighborhood of z_0 we have that $f(z) - f(z_0) = a_m(z - z_0)^m + \sum_{n=m+1}^{+\infty} a_n(z - z_0)^n$ where $a_m \neq 0$ and $m > 1$ (we know that at least $a_1 = f'(z_0) = 0$).

Hence, for $r > 0$ small enough, we have that $\forall z \in D_r(z_0), \left| \sum_{n=m+1}^{+\infty} a_n(z - z_0)^n \right| < |a_m(z - z_0)^m|$.

Then, by Rouché's theorem, the functions $f(z) - f(z_0)$ and $a_m(z - z_0)^m$ have the same number of zeroes on $D_r(z_0)$ (counted with multiplicities), namely $m > 0$.

Hence there exists $z \in D_r(z_0)$ such that $f(z) - f(z_0) = 0$, i.e. $f(z) = f(z_0)$. ■

Remark 9. Once again, this result is false for \mathbb{R} -differentiability:

Indeed $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^3$ is \mathbb{R} -differentiable and injective, but $f'(0) = 0$.

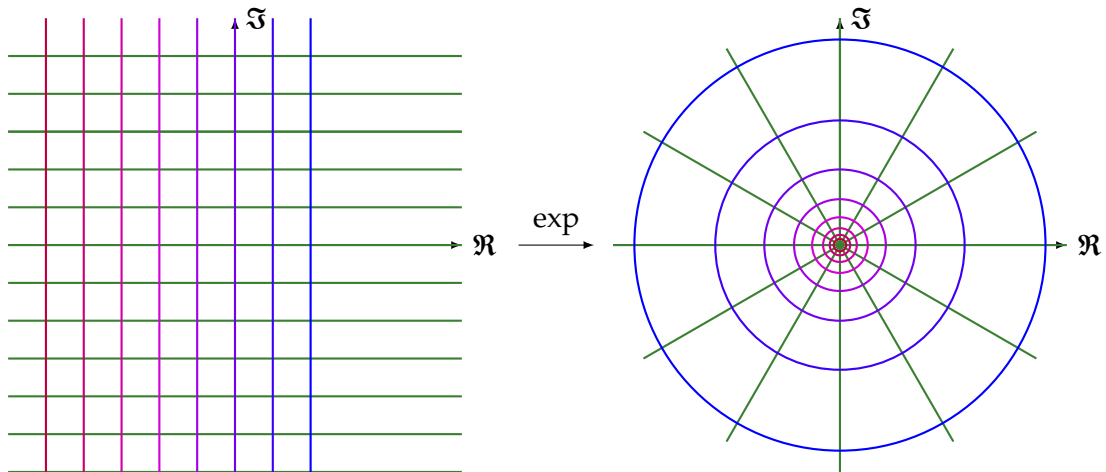
Corollary 10. Let $U \subset \mathbb{C}$ be open. Let $f : U \rightarrow \mathbb{C}$ be holomorphic/analytic.

If f is injective on U then f is conformal on U .

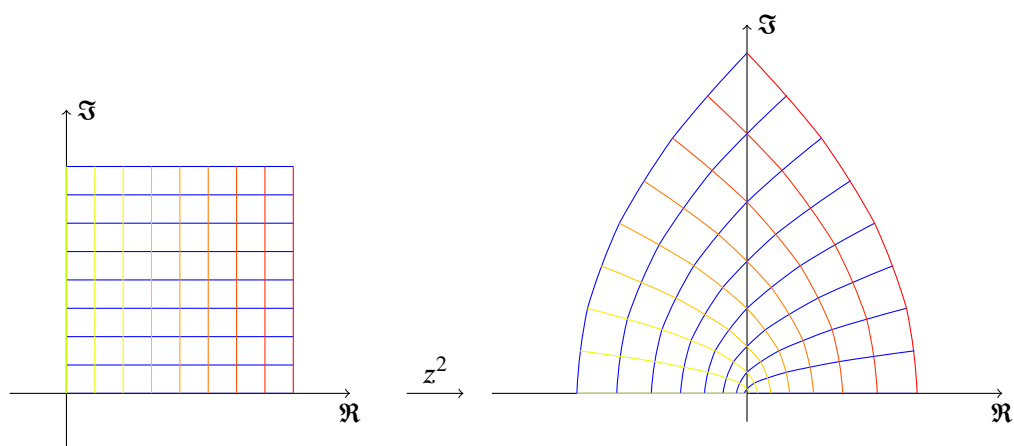
Proof. Let $z_0 \in U$. Since f is holomorphic and injective, we know from the previous that $f'(z_0) \neq 0$. Hence f is conformal at z_0 . ■

4 Examples

1.

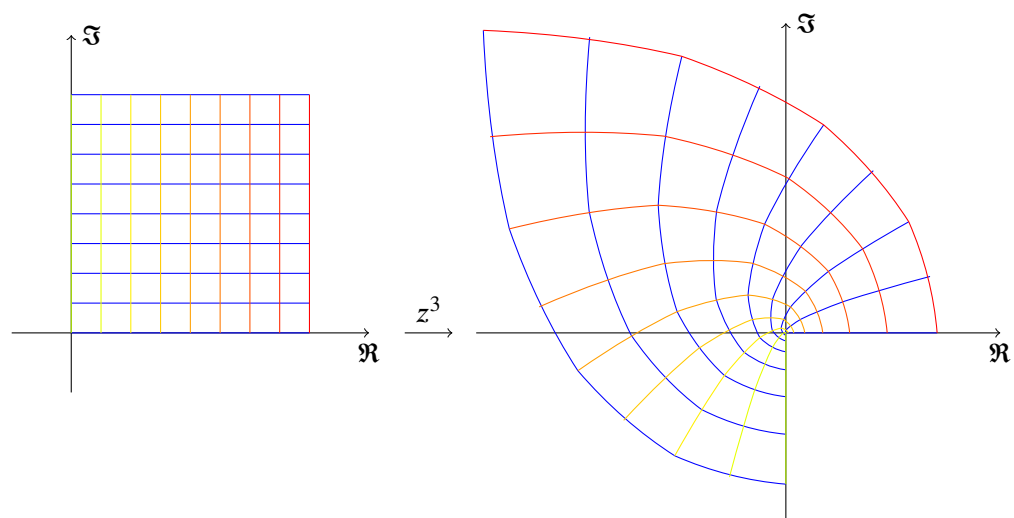


2.



Note that $z \mapsto z^2$ is not conformal at 0.

3.



Note that $z \mapsto z^3$ is not conformal at 0.

4.

