University of Toronto – MAT334H1-F – LEC0101

Complex Variables

11 - Isolated singularities

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Definitions 1. Let $U \subset \mathbb{C}$ be open and $z_0 \in U$. Assume that $f : U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic/analytic.

- Either f is bounded in a neighborhood of z_0,

$$\text{i.e. } \exists M \in \mathbb{R},\ \exists r > 0,\ \forall z \in D_r(z_0) \cap U,\ |f(z)| \leq M,$$

then we say that z_0 is a **removable singularity** of f,

- or $\lim_{z \to z_0} |f(z)| = +\infty$, then we say that z_0 is a **pole** of f,
- otherwise, if none of the above occurs, we say that z_0 is an **essential singularity** of f.

Examples 2 (Removable singularities)**.**

- $f(z) = \frac{z+i}{z^2+1}$ on $\mathbb{C} \setminus \{0\}$ with $z_0 = 0$.
- $f(z) = \frac{\sin z}{z}$ on $\mathbb{C} \setminus \{0\}$ with $z_0 = 0$.

Examples 3 (Poles)**.**

- $f(z) = \frac{1}{z}$ on $\mathbb{C} \setminus \{0\}$ with $z_0 = 0$.
- $f(z) = \frac{1}{z^2-1}$ on $D_1(1)$ with $z_0 = 1$.

Examples 4 (Essential singularities)**.**

- $f(z) = \cos \frac{1}{z}$ on $\mathbb{C} \setminus \{0\}$ with $z_0 = 0$.
- $f(z) = e^{\frac{1}{z}}$ on $\mathbb{C} \setminus \{0\}$ with $z_0 = 0$.

Theorem 5 (Theorem: Riemann's removable singularity theorem)**.**
Let $U \subset \mathbb{C}$ be open and $z_0 \in U$. Assume that $f : U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic/analytic. Then TFAE:

1. *z_0 is a removable singularity of f* (i.e. f is bounded in a neighborhood of z_0)
2. $\lim_{z \to z_0} (z - z_0) f(z) = 0$
3. *f can be holomorphically/analytically extended on U*
   (i.e. there exists $\tilde{f} : U \to \mathbb{C}$ holomorphic/analytic such that $\tilde{f}_{|U \setminus \{z_0\}} = f$)
4. *f can be continuously extended on U*
   (i.e. there exists $\tilde{f} : U \to \mathbb{C}$ continuous such that $\tilde{f}_{|U \setminus \{z_0\}} = f$)

Proof. (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (1). The only non-trivial part is (2) $\implies$ (3).
Define $g : U \to \mathbb{C}$ by $g(z_0) = 0$ and $g(z) = (z - z_0)^2 f(z)$ otherwise. Then

$$\lim_{z\to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z\to z_0} (z - z_0) f(z) = 0$$

Hence g is holomorphic on U.
Since $g(z_0) = g'(z_0) = 0$ we may write

$$g(z) = \sum_{n=2}^{+\infty} a_n (z - z_0)^n = (z - z_0)^2 \sum_{n=2}^{+\infty} a_n (z - z_0)^{n-2}$$

where $z \in D_r(z_0) \cap U$ for some $r > 0$.
Hence $\tilde{f}(z) = \sum_{n=2}^{+\infty} a_n (z - z_0)^{n-2}$ is a suitable holomorphic extension of f on $D_r(z_0) \cap U$. ■

Remark 6. Note that if f admits a continuous extension at z_0 then it is holomorphic.

Examples 7.

- $f(z) = \frac{z^2+1}{z+i}$ on $\mathbb{C} \setminus \{-i\}$ may be holomorphically extended to $\mathbb{C}$ by $\tilde{f}(z) = z - i$.
- $f(z) = \frac{\sin z}{z}$ on $\mathbb{C} \setminus \{0\}$ may be continuous extended at 0 by setting $\tilde{f}(0) = 1$.

Theorem 8. *Let $U \subset \mathbb{C}$ be open and $z_0 \in U$. Assume that $f : U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic/analytic. Then the followings are equivalent:*

1. *z_0 is a pole of f, i.e. $\lim_{z\to z_0} |f(z)| = +\infty$.*
2. *There exist $n \in \mathbb{N}_{>0}$ and $g : U \to \mathbb{C}$ analytic such that $g(z_0) \neq 0$ and $f(z) = \dfrac{g(z)}{(z - z_0)^n}$ on $U \setminus \{z_0\}$.*
3. *z_0 is not a removable singularity of f and there exists $n \in \mathbb{N}_{>0}$ such that $\lim_{z\to z_0} (z - z_0)^{n+1} f(z) = 0$.*

Proof. • (1) $\implies$ (3):
Then $\lim_{z\to z_0} \dfrac{1}{f(z)} = 0$ and z_0 is a removable singularity of $1/f$, so that we can extend it to a holomorphic function $h : U \to \mathbb{C}$ defined by $h(z) = \begin{cases} \frac{1}{f(z)} & \text{if } z \neq z_0 \\ 0 & \text{otherwise} \end{cases}$ (See Remark 6).
Denote by $n := m_h(z_0) \in \mathbb{N}_{>0}$ the order of vanishing of h at z_0 (since $h(z_0) = 0$), then $h(z) = (z - z_0)^n \tilde{h}(z)$ where $\tilde{h} : U \to \mathbb{C}$ is holomorphic and $\tilde{h}(z_0) \neq 0$.
Then $\lim_{z\to z_0} (z - z_0)^{n+1} f(z) = \lim_{z\to z_0} \dfrac{z - z_0}{\tilde{h}(z)} = 0$.

• (3) $\implies$ (2):
Pick the smallest n such that $\lim_{z\to z_0} (z - z_0)^{n+1} f(z) = 0$.
Define $g : U \setminus \{z_0\} \to \mathbb{C}$ by $g(z) = (z - z_0)^n f(z)$. Then $\lim_{z\to z_0} (z - z_0) g(z) = 0$ and z_0 is a removable singularity of g, so that g may be extended to a holomorphic function $g : U \to \mathbb{C}$ by Theorem 5.
Besides $g(z_0) = \lim_{z\to z_0} g(z) = \lim_{z\to z_0} (z - z_0)^n f(z) \neq 0$ by definition of n.

• (2) $\implies$ (1)
Obvious.

■

Definition 9. The integer $n \in \mathbb{N}_{>0}$ in (2) is uniquely defined and we say that f admits a **pole of order n at z_0**.
We saw in the previous proof that the order of the pole z_0 is also:

- The order of vanishing of $1/f$ at z_0.
- The smallest n such that $\lim\limits_{z \to z_0} (z-z_0)^{n+1} f(z) = 0$.

Examples 10.

- 1 is a pole of order 1 of $f(z) = \frac{1}{z^2-1} = \frac{\frac{1}{z+1}}{z-1}$
- 0 is a pole of order 3 of $f(z) = \frac{1}{z^3}$

Theorem 11 (Great Picard's Theorem – **Not part of MAT334**).
Let $U \subset \mathbb{C}$ be open, $z_0 \in U$ and $f : U \setminus \{z_0\} \to \mathbb{C}$ be holomorphic/analytic.
If z_0 is an essential singularity of f then, on any punctured neighborhood of z_0, f takes all possible complex values, with at most a single exception, infinitely many times.

Example 12. $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by $f(z) = e^{\frac{1}{z}}$ has an essential singularity at 0.
It takes the value $w \in \mathbb{C} \setminus \{0\}$ at $z = \frac{1}{\operatorname{Log}(w) + 2i\pi n}$, $n \in \mathbb{Z}$.

Remark 13. In the above proofs, we used in an essential manner that the function f was holomorphic in a punctured neighborhood of z_0, i.e. that there exists $r > 0$ such that f is holomorphic on

$$D_r(z_0) \setminus \{z_0\} = \left\{ z \in \mathbb{C} \ : \ 0 < |z - z_0| < r \right\}$$

Therefore, when we will work with functions having several singularities, we will need to assume that they are **isolated**.

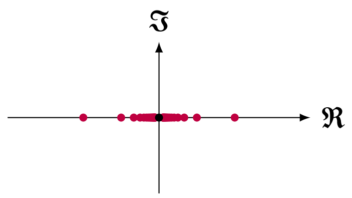
Formally, let $U \subset \mathbb{C}$ be open, $S \subset U$ be the *singular locus* and $f : U \setminus S \to \mathbb{C}$ be holomorphic.
We need that if $z_0 \in S$ then f is holomorphic on $D_r(z_0) \setminus \{z_0\}$ for a small $r > 0$.
Otherwise stated, that there exists a small disk centered at z_0 which doesn't contain another singular point.
To summarize, S needs to satisfy $\forall z_0 \in S$, $\exists r > 0$, $D_r(z_0) \cap S = \{z_0\}$.

If you take MAT327, it simply means that S is discrete in U.

We will only study isolated singularities, we won't study wilder singular loci.

Examples 14.

- The function $f : \mathbb{C} \setminus \{\pm 1\} \to \mathbb{C}$ defined by $f(z) = \frac{1}{z^2-1}$ is holomorphic and has 2 isolated singularities at -1 and $+1$.
- Let $f : \mathbb{C} \setminus \left(\left\{ \frac{1}{\pi n}, n \in \mathbb{Z} \right\} \cup \{0\} \right) \to \mathbb{C}$ be defined by $f(z) = \cot \frac{1}{z}$.
Then 0 is not an isolated singularity of f:



Definitions 15. Let $U \subset \mathbb{C}$ be an open neighborhood of infinity, i.e. there exists $r > 0$ such that $\{z \in \mathbb{C} : |z| > r\} \subset U$. Let $f : U \to \mathbb{C}$ be holomorphic/analytic.
Then ∞ is an isolated singularity of f (i.e. f is defined in a neighborhood of ∞ but not at ∞).

1. Either f is bounded in a neighborhood of ∞,

$$\text{i.e. } \exists M \in \mathbb{R},\, \exists r > 0,\, \forall z \in U,\, |z| > r \implies |f(z)| \leq M,$$

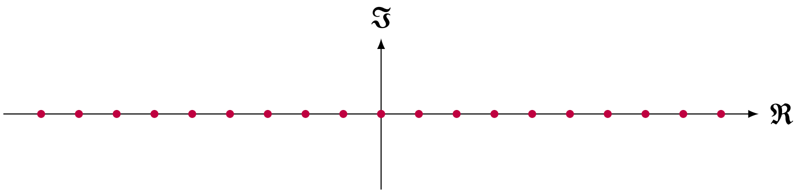
   then we say that ∞ is a **removable singularity** of f,

2. or $\lim_{z\to\infty} |f(z)| = +\infty$, then we say that ∞ is a **pole** of f,

3. otherwise, if none of the above occurs, we say that ∞ is an **essential singularity** of f.

Remark 16. Recall that the inversion $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, $z \mapsto \frac{1}{z}$, swaps 0 and ∞.
Hence, if we set $g(z) = f\left(\frac{1}{z}\right)$ then the type of singularity of f at ∞ coincides with the type of singularity of g at 0.

Example 17. Let $f : \mathbb{C} \setminus \{\pi n : n \in \mathbb{Z}\} \to \mathbb{C}$ be defined by $f(z) = \cot z$.
Then ∞ is not an isolated singularity of f:



Spoiler: a first introduction to Laurent series
Assume that z_0 is a pole of order $n \in \mathbb{N}_{>0}$ of f.
Then there exists $g : U \to \mathbb{C}$ holomorphic/analytic such that $g(z_0) \neq 0$ and $f(z) = \dfrac{g(z)}{(z-z_0)^n}$ in a neighborhood of z_0.
Since g is analytic at z_0, it may be expressed as a power series in a small neighborhood of z_0:

$$g(z) = \sum_{k=0}^{+\infty} a_k (z-z_0)^k$$

and since $g(z_0) \neq 0$ we know that $a_0 \neq 0$.
Therefore, in a punctured neighborhood of z_0, we may express f as

$$\begin{aligned} f(z) &= \sum_{k\geq -n}^{+\infty} a_{k+n}(z-z_0)^k \\ &= a_0(z-z_0)^{-n} + a_1(z-z_0)^{-n+1} + \cdots + a_n + a_{n+1}(z-z_0) + a_{n+2}(z-z_0)^2 + \cdots \end{aligned}$$

Note that the above expression has some negative exponents: it is a first example of *Laurent series*, notion that we will study next week.