

CAUCHY'S RESIDUE THEOREM



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Cauchy's residue theorem – 1

Theorem: the residue theorem

Let $U \subset \mathbb{C}$ be open. Let $S \subset U$ be finite. Assume that $f : U \setminus S \rightarrow \mathbb{C}$ is holomorphic/analytic.

Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a positively oriented piecewise smooth simple closed curve on $U \setminus S^a$ whose inside^b is entirely included in U . Then^c

$$\int_{\gamma} f(z) dz = 2i\pi \sum_{z \in \text{Inside}(\gamma)} \text{Res}(f, z)$$

^ai.e. γ doesn't pass through any point of S .

^bSee Jordan's curve theorem, September 28.

^cThe following sum is finite since $\text{Res}(f, z) \neq 0$ only for $z \in S$.

Corollary

Let $U \subset \mathbb{C}$ be open and simply-connected. Let $S \subset U$ be finite.

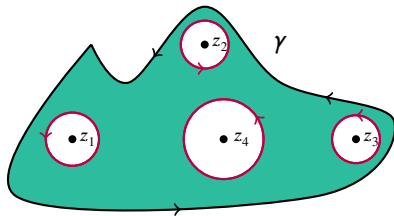
Assume that $f : U \setminus S \rightarrow \mathbb{C}$ is holomorphic/analytic.

Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a positively oriented piecewise smooth simple closed curve on $U \setminus S$. Then

$$\int_{\gamma} f(z) dz = 2i\pi \sum_{z \in \text{Inside}(\gamma)} \text{Res}(f, z)$$

Cauchy's residue theorem – 2

Proof.



We may find pairwise disjoint disks $\overline{D_{r_k}(z_k)} \subset U$ where $\{z_1, \dots, z_n\}$ are the points of S enclosed in γ .

We apply Green's theorem to $T = \overline{\text{Inside}(\gamma)} \setminus \left(\bigcup_{i=1}^n D_{r_k}(z_k) \right)$, then

$$\int_{\gamma} f(z) dz - \sum_{k=1}^n \int_{\gamma_k} f(z) dz = i \iint_T \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) = 0$$

where $\gamma_k : [0, 1] \rightarrow \mathbb{C}$ is defined by $\gamma_k(t) = z_k + r_k e^{2i\pi t}$.

The last equality is due to the Cauchy–Riemann equations.

$$\text{Then } \int_{\gamma} f(z) dz = \sum_{k=1}^n \int_{\gamma_k} f(z) dz = 2i\pi \sum_{k=1}^n \text{Res}(f, z_k)$$

Cauchy's residue theorem – 3

Corollary

Let $S \subset \mathbb{C}$ be finite. Assume that $f : \mathbb{C} \setminus S \rightarrow \mathbb{C}$ is holomorphic/analytic. Then

$$\operatorname{Res}(f, \infty) + \sum_{z \in S} \operatorname{Res}(f, z) = 0$$

Proof. Take $r > 0$ such that $S \subset D_r(0)$ and define $\gamma : [0, 1] \rightarrow \mathbb{C}$ by $\gamma(t) = re^{2i\pi t}$. Then

$$\begin{aligned} \sum_{z \in S} \operatorname{Res}(f, z) &= \frac{1}{2i\pi} \int_{\gamma} f(z) dz \quad \text{by Cauchy's residue theorem} \\ &= -\operatorname{Res}(f, \infty) \end{aligned}$$



We may rewrite the above conclusion as $\sum_{z \in \hat{\mathbb{C}}} \operatorname{Res}(f, z) = 0$.

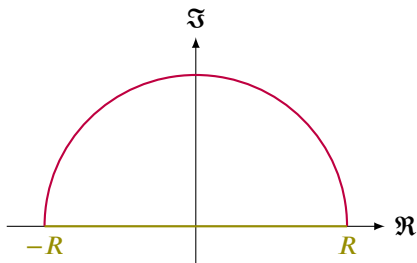
Example: $\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} dx$ (rational function) – 1

Assume that P and Q are two polynomials. Set $f(z) = \frac{P(z)}{Q(z)}$.

We know that $\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} dx$ is convergent if and only if $\deg Q \geq \deg P + 2$, let's assume the latter.

Then $\lim_{|z| \rightarrow \infty} zf(z) = 0$.

Assume that Q has no real root.



Define $\gamma_R : [0, 1] \rightarrow \mathbb{C}$ by $\gamma_R(t) = Re^{i\pi t}$.

For $R > 0$ big enough, all the poles of f whose imaginary part is positive are included within the upper-half disk centered at 0 and of radius R .

$$\text{Then } \int_{-R}^R f(z) dz + \int_{\gamma_R} f(z) dz = 2i\pi \sum_{z \text{ s.t. } \Im(z) > 0} \text{Res}(f, z).$$

$$\text{But } \left| \int_{\gamma_R} f \right| \leq \pi R \sup_{\gamma_R} |f| \xrightarrow{R \rightarrow +\infty} 0 \text{ since } \lim_{|z| \rightarrow \infty} zf(z) = 0.$$

$$\text{Hence } \int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} dx = 2i\pi \sum_{z \text{ s.t. } \Im(z) > 0} \text{Res} \left(\frac{P}{Q}, z \right).$$

Example: $\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} dx$ (rational function) – 2

Example

Compute $\int_{-\infty}^{+\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)}$ où $0 < a < b$.

The poles of $f(z) = \frac{1}{(z^2 + a^2)(z^2 + b^2)}$ are $-ia, ia, -ib$ and ib which are simple. Hence

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)} &= 2i\pi \operatorname{Res}(f, ia) + 2i\pi \operatorname{Res}(f, ib) \\ &= \frac{2i\pi}{2i(b^2 - a^2)a} + \frac{2i\pi}{2i(a^2 - b^2)b} \\ &= \frac{\pi}{ab(a + b)} \end{aligned}$$

Example: $\int_0^{2\pi} R(\cos t, \sin t) dt$

Set $z = e^{it}$ then $\cos(t) = \frac{1}{2} \left(z + \frac{1}{z} \right)$, $\sin(t) = \frac{1}{2i} \left(z - \frac{1}{z} \right)$, and $\frac{dz}{iz} = dt$.

Set $f(z) = \frac{1}{iz} R\left(\frac{1}{2} \left(z + \frac{1}{z} \right), \frac{1}{2i} \left(z - \frac{1}{z} \right)\right)$. Then $\int_0^{2\pi} R(\cos t, \sin t) dt = \int_{S^1} f(z) dz = 2i\pi \sum_{z \in D_1(0)} \text{Res}(f, z)$.

Example

Compute $\int_0^{2\pi} \frac{a}{a^2 + \sin^2 t} dt$ where $a > 0$.

Set $f(z) = \frac{1}{iz} \frac{a}{a^2 - \frac{1}{4} \left(z - \frac{1}{z} \right)^2} = -\frac{4iaz}{(z^2 + 2az - 1)(z^2 - 2az - 1)}$.

Note that the singularity at 0 of the LHS is removable since we may extend f through 0 using the RHS, so that $\text{Res}(f, 0) = 0$.

Then, the only poles of f within the unit disk are $z_1 = -a + \sqrt{a^2 + 1}$ and $z_2 = a - \sqrt{a^2 + 1}$ which are simple. Hence

$$\int_0^{2\pi} \frac{a}{a^2 + \sin^2 t} dt = 2i\pi \text{Res}(f, z_1) + 2i\pi \text{Res}(f, z_2) = \frac{2\pi}{\sqrt{a^2 + 1}}$$

Jordan's lemma

The following trick called, *Jordan's lemma*, can be very useful.

Let $\gamma_R : [0, \pi] \rightarrow \mathbb{C}$ be defined by $\gamma_R(t) = Re^{it}$.

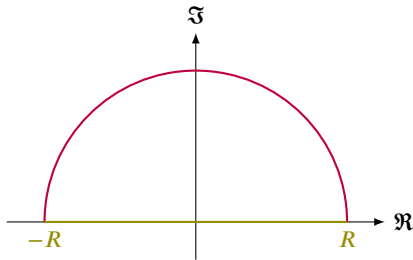
$$\begin{aligned} \left| \int_{\gamma_R} g(z) e^{iz} dz \right| &= \left| \int_0^\pi g(Re^{it}) e^{iR(\cos t + i \sin t)} i Re^{it} dt \right| \\ &\leq \int_0^\pi |g(Re^{it}) e^{iR(\cos t + i \sin t)} i Re^{it}| dt \\ &\leq R \sup_{\gamma_R} |g| \int_0^\pi e^{-R \sin t} dt \\ &= 2R \sup_{\gamma_R} |g| \int_0^{\pi/2} e^{-R \sin t} dt \\ &\leq 2R \sup_{\gamma_R} |g| \int_0^{\pi/2} e^{-2Rt/\pi} dt \quad \text{by Jordan's inequality: } \forall x \in \left[0, \frac{\pi}{2}\right], \frac{2}{\pi}x \leq \sin(x) \leq x \\ &\leq \pi \sup_{\gamma_R} |g| (1 - e^{-R}) \\ &\leq \pi \sup_{\gamma_R} |g| \end{aligned}$$

Example: $\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} e^{iax} dx - 1$

We want to compute $I(a) := \int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} e^{iax} dx$ where P, Q are real polynomials and $a \in \mathbb{R}$. Assume that Q has no real root.

Note that $I(-a) = \overline{I(a)}$, so we may restrict our attention to $a > 0$.

The integral is convergent if and only if $\deg Q \geq \deg P + 1$ (integration by parts), so we assume the latter.



Define $\gamma_R : [0, 1] \rightarrow \mathbb{C}$ by $\gamma_R(t) = Re^{i\pi t}$.

For $R > 0$ big enough, all the poles of $f(z) = \frac{P(z)}{Q(z)} e^{iaz}$ whose imaginary part is positive are included within the upper-half disk centered at 0 and of radius R .

$$\text{Then } \int_{-R}^R f(z) dz + \int_{\gamma_R} f(z) dz = 2i\pi \sum_{z \text{ s.t. } \Im(z) > 0} \text{Res}(f, z) \quad (*)$$

But, by Jordan's lemma,

$$\left| \int_{\gamma_R} \frac{P(z)}{Q(z)} e^{iz} dz \right| \leq \pi \sup_{\gamma_R} |P/Q| \xrightarrow{R \rightarrow +\infty} 0$$

Hence, by taking $R \rightarrow +\infty$ in $(*)$, we get

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} e^{iax} dx = 2i\pi \sum_{z \text{ s.t. } \Im(z) > 0} \text{Res} \left(\frac{P(z)}{Q(z)} e^{iaz}, z \right)$$

Example: $\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} e^{iax} dx - 2$

Example

Compute $\int_{-\infty}^{+\infty} \frac{\cos(x)}{x^2 + \alpha^2} dx$ where $\alpha > 0$.

Note that $\int_{-\infty}^{+\infty} \frac{\cos(x)}{x^2 + \alpha^2} dx = \Re \left(\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2 + \alpha^2} dx \right)$.

The poles of $f(z) = \frac{e^{iz}}{z^2 + \alpha^2}$ are $i\alpha$ and $-i\alpha$ which are simple.

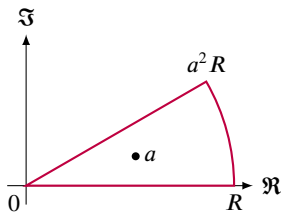
By the previous slide, $\int_{-\infty}^{+\infty} \frac{e^{iz}}{z^2 + \alpha^2} dz = 2i\pi \operatorname{Res}(f, i\alpha) = 2i\pi \frac{e^{-\alpha}}{2i\alpha} = \pi \frac{e^{-\alpha}}{\alpha}$.

Hence $\int_{-\infty}^{+\infty} \frac{\cos(x)}{x^2 + \alpha^2} dx = \pi \frac{e^{-\alpha}}{\alpha}$.

Example: $\int_0^{+\infty} \frac{x^p}{1+x^n} dx$, $n, p \in \mathbb{N}$

We know that the integral $\int_0^{+\infty} \frac{x^p}{1+x^n} dx$ is convergent if and only if $n \geq p+2$. Set $f(z) = \frac{z^p}{1+z^n}$ and $a = e^{i\frac{\pi}{n}}$.

We consider the following sector of the circle centered at 0 and of radius R , such that the only pole of f enclosed in its inside is a .



Let $\gamma : [0, \frac{2\pi}{n}] \rightarrow \mathbb{C}$ be defined by $\gamma(t) = Re^{it}$.

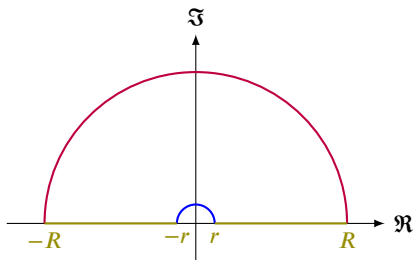
By the residue theorem, $2i\pi \operatorname{Res}(f, a) = \int_{[0,R]} f + \int_{\gamma} f + \int_{[a^2 R, 0]} f$.

- $\operatorname{Res}(f, a) = \frac{a^p}{na^{n-1}} = -\frac{a^{p+1}}{n}$.
- $\int_{[a^2 R, 0]} f(z) dz = -a^2 \int_0^R \frac{a^{2p} t^p}{1 + a^{2n} t^n} dt = -a^{2(p+1)} \int_0^R \frac{t^p}{1 + t^n} dt$
- $\left| \int_{\gamma} f(z) dz \right| \leq \frac{2\pi}{n} R \sup_{\gamma} |f| \xrightarrow{R \rightarrow +\infty} 0$ since $\lim_{z \rightarrow \infty} z f(z) = 0$.

Hence, by taking the limit as $R \rightarrow +\infty$, we get $-2i\pi \frac{a^{p+1}}{n} = \int_0^{+\infty} \frac{x^p}{1+x^n} dx - a^{2(p+1)} \int_0^{+\infty} \frac{x^p}{1+x^n} dx$.

Finally $\int_0^{+\infty} \frac{x^p}{1+x^n} dx = \frac{2i\pi}{n} \frac{a^{p+1}}{a^{2(p+1)} - 1} = \frac{\pi}{n} \frac{2i}{a^{p+1} - a^{-(p+1)}} = \frac{\pi}{n \sin \frac{(p+1)\pi}{n}}$.

Example: $\int_0^{+\infty} \frac{\sin t}{t} dt$



Define $\gamma_R : [0, 1] \rightarrow \mathbb{C}$ by $\gamma_R(t) = Re^{i\pi t}$ and $\gamma_r : [0, 1] \rightarrow \mathbb{C}$ by $\gamma_r(t) = re^{i\pi t}$.

By Cauchy's integral theorem

$$\int_{\gamma_R} \frac{e^{iz}}{z} dz - \int_{\gamma_r} \frac{e^{iz}}{z} dz + \int_{-R}^{-r} \frac{e^{it}}{t} dt + \int_r^R \frac{e^{it}}{t} dt = 0.$$

- By Jordan's lemma: $\left| \int_{\gamma_R} \frac{e^{iz}}{z} dz \right| \leq \pi \sup_{\gamma_R} |1/z| \xrightarrow{R \rightarrow +\infty} 0$

- Since 0 is a simple pole of $f(z) = \frac{e^{iz}}{z}$, we have that $f(z) = \text{Res}(f, 0)z^{-1} + g(z)$ where g is holomorphic.

Then $\int_{\gamma_r} f(z) dz = \int_{\gamma_r} \text{Res}(f, 0)z^{-1} dz + \int_{\gamma_r} g(z) dz$ but

$$\int_{\gamma_r} \text{Res}(f, 0)z^{-1} dz = \text{Res}(f, 0)i\pi \text{ and } \left| \int_{\gamma_r} g(z) dz \right| \leq \pi r \sup_{\gamma_r} |g| \xrightarrow{r \rightarrow 0} 0.$$

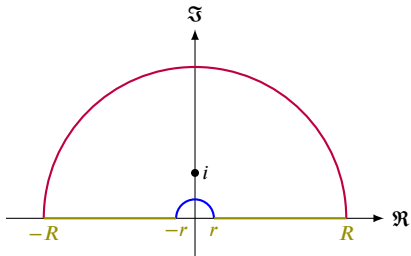
Hence $\int_{\gamma_r} f(z) dz \xrightarrow{r \rightarrow 0} \text{Res}(f, 0)i\pi = i\pi$.

$$\int_r^R \frac{\sin t}{t} dt = \frac{1}{2i} \int_r^R \frac{e^{it} - e^{-it}}{t} dt = \frac{1}{2i} \int_r^R \frac{e^{it}}{t} dt - \frac{1}{2i} \int_r^R \frac{e^{-it}}{t} dt = \frac{1}{2i} \int_r^R \frac{e^{it}}{t} dt + \frac{1}{2i} \int_{-R}^{-r} \frac{e^{it}}{t} dt = \frac{1}{2i} \left(\int_{\gamma_r} \frac{e^{iz}}{z} dz - \int_{\gamma_R} \frac{e^{iz}}{z} dz \right)$$

Hence, taking $r \rightarrow 0$ and $R \rightarrow +\infty$ we get that $\int_0^{+\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$

Example: $\int_0^{+\infty} \frac{(\log t)^2}{1+t^2} dt$

We set $f(z) = \frac{(\log z)^2}{1+z^2}$ but for that we need to fix a branch of the logarithm. Let's fix $\log : \mathbb{C} \setminus \{iy : y \leq 0\} \rightarrow \mathbb{C}$ defined by $\log z = \log |z| + i \operatorname{Arg}(z)$ where $\operatorname{Arg}(z) \in \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right)$.



Define $\gamma_R : [0, 1] \rightarrow \mathbb{C}$ by $\gamma_R(t) = Re^{i\pi t}$ and $\gamma_r : [0, 1] \rightarrow \mathbb{C}$ by $\gamma_r(t) = re^{i\pi t}$.

By Cauchy's residue theorem

$$\int_{\gamma_R} f(z) dz - \int_{\gamma_r} f(z) dz + \int_{-R}^{-r} f(z) dz + \int_r^R f(z) dz = 2i\pi \operatorname{Res}(f, i) = -\frac{\pi^3}{4}.$$

- Since $|\log z| \leq |\ln r| + \pi$ on γ_r , $\int_{\gamma_r} f(z) dz \leq \pi r \frac{(|\ln r| + \pi)^2}{1+r^2} \xrightarrow{r \rightarrow +\infty \text{ or } 0} 0$.

- Since $z = te^{i\pi}$ on $[-R, -r]$, we have

$$\int_{-R}^{-r} f(z) dz = \int_r^R \frac{(\ln t + i\pi)^2}{1+t^2} dt = \int_r^R \frac{(\ln t)^2}{1+t^2} dt + 2i\pi \int_r^R \frac{\ln t}{1+t^2} dt - \int_r^R \frac{\pi^2}{1+t^2} dt$$

By taking the limits $r \rightarrow 0$ and $R \rightarrow +\infty$ we get $\int_0^{+\infty} \frac{(\ln t)^2}{1+t^2} dt + 2i\pi \int_0^{+\infty} \frac{\ln t}{1+t^2} dt - \frac{\pi^3}{2} + \int_0^{+\infty} \frac{(\ln t)^2}{1+t^2} dt = -\frac{\pi^3}{4}$.

By taking the real part, we get $\int_0^{+\infty} \frac{(\ln t)^2}{1+t^2} dt = \frac{\pi^3}{8}$.