Laurent series and Residues

October 26th, 2020 and October 28th, 2020

Informal presentation

Recall that a function $f : U \to \mathbb{C}$ ($U \subset \mathbb{C}$ open) is holomorphic/analytic if and only if f can be described as a power series

$$f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n$$

in a neighborhood of every point $z_0 \in U$.
We are going to generalize this property in order to study f in the neighborhood of an isolated singularity z_0 (i.e. f is not defined at z_0 but is holomorphic in a punctured neighborhood of z_0), for that purpose we are going to work with Laurent series allowing negative exponents:

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n$$

Intuitively, the more we need negative exponents in the above expression, the wilder is the singularity.

Laurent's theorem

Theorem: Laurent's theorem

Let $z_0 \in \mathbb{C}$ and $0 \leq r < R \leq +\infty$. Set $U = \left\{z \in \mathbb{C} \ : \ r < |z - z_0| < R\right\}$ and let $f : U \to \mathbb{C}$ be holomorphic/analytic then

$$\forall z \in U,\ f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where $a_n = \dfrac{1}{2i\pi} \displaystyle\int_\gamma \dfrac{f(w)}{(w - z_0)^{n+1}} \mathrm{d}w$ and $\gamma : [0, 1] \to \mathbb{C}$ is defined by $\gamma(t) = z_0 + \rho e^{2i\pi t}$ with $\rho \in (r, R)$.

We call such a series (a *"power series"* with exponents in $\mathbb{Z}$) a **Laurent series**.

- If $R = +\infty$ then U is the complement of the closed disk centered at z_0 and of radius r, hence it is a neighborhood of ∞.
- If $r = 0$ then U is a punctured open disk centered at z_0 and of radius R.
- If $r = 0$ and $R = +\infty$ then $U = \mathbb{C} \setminus \{z_0\}$.

Example:

The function $f : \mathbb{C} \setminus \{0,1\} \to \mathbb{C}$ defined by $f(z) = \frac{1}{z(1-z)}$ is holomorphic on $\mathbb{C} \setminus \{0,1\}$ and has two isolated singularities, namely 0 and 1.

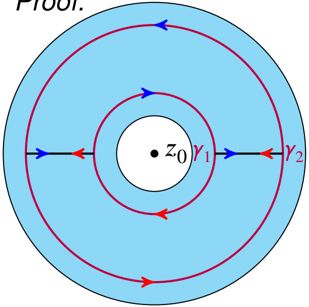
- For $0 < |z| < 1$, $f(z) = \dfrac{1}{z} + \dfrac{1}{1-z} = z^{-1} + \displaystyle\sum_{n=0}^{+\infty} z^n = \sum_{n=-1}^{+\infty} z^n$.
- For $0 < |z-1| < 1$, $f(z) = -\dfrac{1}{z-1} + \dfrac{1}{z} = -(z-1)^{-1} + \displaystyle\sum_{n=0}^{+\infty}(-1)^n(z-1)^n = \sum_{n=-1}^{+\infty}(-1)^n(z-1)^n$.
- For $1 < |z|$, $f(z) = -\dfrac{1}{z^2}\dfrac{z}{z-1} = -\dfrac{1}{z^2}\dfrac{1}{1-\frac{1}{z}} = -\dfrac{1}{z^2}\displaystyle\sum_{n=0}^{+\infty} z^{-n} = \sum_{n=-\infty}^{-2}(-1)z^n$

Proof of Laurent's theorem – 1

Lemma

Let $z_0 \in \mathbb{C}$ and $0 \leq r < R \leq +\infty$. Set $U = \left\{z \in \mathbb{C} \,:\, r < |z - z_0| < R\right\}$ and let $g : U \to \mathbb{C}$ be holomorphic/analytic.
Let $\rho_1, \rho_2 \in \mathbb{R}$ be s.t. $r < \rho_1 < \rho_2 < R$. Let $\gamma_1, \gamma_2 : [0,1] \to \mathbb{C}$ be defined by $\gamma_k(t) = z_0 + \rho_k e^{2i\pi t}$.
Then $\displaystyle\int_{\gamma_1} g(z)\mathrm{d}z = \int_{\gamma_2} g(z)\mathrm{d}z$.

Proof.

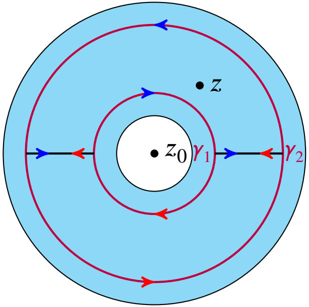


We consider two simple closed curves as in the drawing (in red and blue), then by Cauchy's integral theorem:

$$\begin{aligned} 0 = 0 + 0 &= \int_{red} g(z)\mathrm{d}z + \int_{blue} g(z)\mathrm{d}z \\ &= \int_{-\gamma_1} g(z)\mathrm{d}z + \int_{\gamma_2} g(z)\mathrm{d}z \\ &= -\int_{\gamma_1} g(z)\mathrm{d}z + \int_{\gamma_2} g(z)\mathrm{d}z \end{aligned}$$

Hence $\displaystyle\int_{\gamma_1} g(z)\mathrm{d}z = \int_{\gamma_2} g(z)\mathrm{d}z$. ∎

Proof of Laurent's theorem.



Let $r < \rho_1 < |z| < \rho_2 < R$ and $\gamma_1, \gamma_2 : [0,1] \to \mathbb{C}$ defined by $\gamma_k(t) = z_0 + \rho_k e^{2i\pi t}$.
We consider two simple closed curves as in the drawing (in red and blue).
Then by Cauchy's integral formula and theorem

$$\begin{aligned}
0 + 2i\pi f(z) &= \int_{red} \frac{f(w)}{w-z}\mathrm{d}w + \int_{blue} \frac{f(w)}{w-z}\mathrm{d}w \\
&= -\int_{\gamma_1} \frac{f(w)}{w-z}\mathrm{d}w + \int_{\gamma_2} \frac{f(w)}{w-z}\mathrm{d}w \\
&= -\int_{\gamma_1} \frac{f(w)}{z_0-z}\frac{1}{1-\frac{w-z_0}{z-z_0}}\mathrm{d}w + \int_{\gamma_2} \frac{f(w)}{w-z_0}\frac{1}{1-\frac{z-z_0}{w-z_0}}\mathrm{d}w \\
&= \int_{\gamma_1} \frac{f(w)}{z_0-z}\sum_{n\geq 0}\left(\frac{w-z_0}{z_0-z}\right)^n\mathrm{d}w + \int_{\gamma_2} \frac{f(w)}{w-z_0}\sum_{n\geq 0}\left(\frac{z-z_0}{w-z_0}\right)^n\mathrm{d}w \\
&= \sum_{n<0}\left(\int_{\gamma_1} \frac{f(w)}{(w-z_0)^{n+1}}\mathrm{d}w\right)(z-z_0)^n + \sum_{n\geq 0}\left(\int_{\gamma_2} \frac{f(w)}{(w-z_0)^{n+1}}\mathrm{d}w\right)(z-z_0)^n
\end{aligned}$$

For $w \in \gamma_1$, $\left|\frac{w-z_0}{z-z_0}\right| < 1$.

For $w \in \gamma_2$, $\left|\frac{z-z_0}{w-z_0}\right| < 1$.

Then we can replace γ_1 and γ_2 by γ thanks to the previous lemma. ■

We need to justify the $\sum - \int$ permutation of the last equality: that's exactly the same proof as for the power expression of a holomorphic function (see lecture from October 16).

Laurent series and isolated singularities – 1

Proposition

Let $U \subset \mathbb{C}$ be open and $z_0 \in U$. Assume that $f : U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic/analytic.
Let $R > 0$ be such that $D_R(z_0) \subset U$. Then f admits a Laurent series expansion centered at z_0 on $D_R(z_0) \setminus \{z_0\}$ (apply Laurent's theorem with $r = 0$)

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Then

1. z_0 is a removable singularity iff $\forall n < 0,\ a_n = 0$
2. z_0 is a pole iff $\exists m \geq 1,\ a_{-m} \neq 0$ and $\forall n < -m,\ a_n = 0$ *(m is the order of the pole z_0)*
3. z_0 is an essential singularity if and only if for infinitely many $n \in \mathbb{N}$, $a_{-n} \neq 0$.

Proof.

1. f coincides on $D_R(z_0) \setminus \{z_0\}$ with $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ holomorphic at z_0.
2. f coincides on $D_R(z_0) \setminus \{z_0\}$ with $\frac{\sum_{n=0}^{\infty} a_{n-m}(z-z_0)^n}{(z-z_0)^m}$.
3. if z_0 is not a removable singularity nor a pole then it is an essential singularity. ∎

Example

For $z \in \mathbb{C} \setminus \{0\}$, $e^{\frac{1}{z}} = \sum_{n=0}^{+\infty} \frac{\left(z^{-1}\right)^n}{n!} = \sum_{n=-\infty}^{0} \frac{z^n}{(-n)!}$.

Hence $e^{\frac{1}{z}}$ has an essential singularity at 0.

Theorem

Let $U \subset \mathbb{C}$ be an open neighborhood of infinity, i.e. there exists $r > 0$ such that $\{z \in \mathbb{C} \ : \ |z| > r\} \subset U$. Let $f : U \to \mathbb{C}$ be holomorphic/analytic.
Then f admits a Laurent series expansion centered at 0 on $\{z \in \mathbb{C} \ : \ |z| > r\}$ (apply Laurent's theorem with $R = +\infty$)

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

Then

1. ∞ is a removable singularity iff $\forall n > 0,\ a_n = 0$
2. ∞ is a pole iff $\exists m \geq 1,\ a_m \neq 0$ and $\forall n > m,\ a_n = 0$ *(m is the order of the pole at ∞)*
3. ∞ is an essential singularity if and only if for infinitely many $n \in \mathbb{N}$, $a_n \neq 0$.

Proof. Recall that the inversion $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}},\ z \mapsto \frac{1}{z}$, swaps 0 and ∞. ■

Principal part

Definition

The **principal part** of a Laurent series

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n$$

is the part consisting only of the negative exponents

$$\sum_{n=-\infty}^{-1} a_n (z - z_0)^n$$

Residues – 1

Definition: residue

Let z_0 be an isolated singularity of f. Denote the Laurent expansion of f around z_0 by

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n$$

Then the **residue of f at z_0** is the coefficient of $(z - z_0)^{-1}$, i.e. $\mathrm{Res}(f, z_0) := a_{-1}$.

Proposition

Assume that f is holomorphic on $D_r(z_0) \setminus \{z_0\}$ for some $r > 0$ then

$$\mathrm{Res}(f, z_0) = \frac{1}{2i\pi} \int_\gamma f(w) \mathrm{d}w$$

where $\gamma : [0, 1] \to \mathbb{C}$ is defined by $\gamma(t) = z_0 + \rho e^{2i\pi t}$ with $\rho \in (0, r)$.

Note that the above integral doesn't depend on the choice of ρ by the above lemma.

Residues – 2

Example

In a punctured neighborhood of 0, we have

$$\frac{e^z-1}{z^4} = \frac{1}{z^3} + \frac{1}{2z^2} + \frac{1}{6z} + \frac{1}{24} + \frac{z}{120} + \cdots$$

Hence $\operatorname{Res}\left(\frac{e^z-1}{z^4}, 0\right) = \frac{1}{6}$.

Residues – 3

Proposition: how to compute residues

- If z_0 is a pole of order 1 of f then $\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$.
- If z_0 is a pole of order k of f then $\operatorname{Res}(f, z_0) = \dfrac{h^{(k-1)}(z_0)}{(k-1)!}$ where $h(z) = (z - z_0)^k f(z)$.
- If f is holomorphic at z_0 and g has a zero of order 1 at z_0 then $\operatorname{Res}\left(\dfrac{f}{g}, z_0\right) = \dfrac{f(z_0)}{g'(z_0)}$.
- Assume that z_0 is an isolated zero of f then the order of vanishing of f at z_0 is $\operatorname{Res}\left(\dfrac{f'}{f}, z_0\right)$.

Homework

- Prove the above identities!
- Compute some residues (i.e. practice exercises from the textbook!).

Residue at ∞

Definition: residue at ∞

Assume that f is holomorphic/analytic in a neighborhood of infinity, i.e. on $\{z \in \mathbb{C} \; : \; |z| > r\}$ for some r. We define the **residue of f at ∞** by

$$\operatorname{Res}(f, \infty) := \operatorname{Res}\left(\frac{-1}{z^2} f\left(\frac{1}{z}\right), 0\right)$$

Proposition

Assume that f is holomorphic on $\{z \in \mathbb{C} \; : \; |z| > r\}$ for some $r > 0$ then

$$\operatorname{Res}(f, \infty) = -\frac{1}{2i\pi} \int_\gamma f(w) \mathrm{d}w$$

where $\gamma : [0, 1] \to \mathbb{C}$ is defined by $\gamma(t) = z_0 + \rho e^{2i\pi t}$ with $\rho > r$.

The sign is due to the fact that γ is not positively oriented: it is considered as the boundary of the complement of the disk since that's where is located ∞.

Homework: prove the proposition (you will see where does $-\frac{1}{z^2}$ come from).