MAT334H1-F – LEC0101

Complex Variables

http://uoft.me/MAT334-LEC0101

LINE INTEGRALS

UNIVERSITY OF TORONTO

September 28$^{\text{th}}$, 2020 and September 30$^{\text{th}}$, 2020

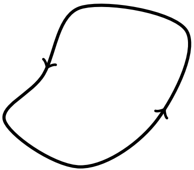
- A (complex) **curve** is a continuous function $\gamma : [a,b] \to \mathbb{C}$ where $a < b$. We say that $\gamma(a)$ is the **start-point** of γ and that $\gamma(b)$ is its **end-point**.
  Careful: it is quite common to designate by "curve" either the function γ or its range $\gamma([a,b])$.

- We say that a curve $\gamma : [a,b] \to \mathbb{C}$ is **simple** if it doesn't admit double points (i.e. self-intersections), except maybe $\gamma(a) = \gamma(b)$, formally

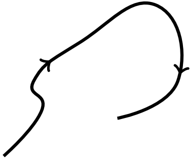
$$\left(a \leq t_1 < t_2 \leq b \text{ and } \gamma(t_1) = \gamma(t_2) \right) \implies \left(t_1 = a \text{ and } t_2 = b \right)$$

- We say that a curve $\gamma : [a,b] \to \mathbb{C}$ is **closed** if $\gamma(a) = \gamma(b)$.

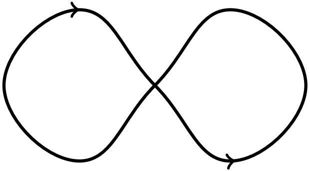
Curves – 2



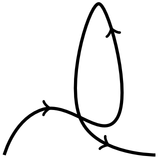
Closed, simple.



Not closed, simple.



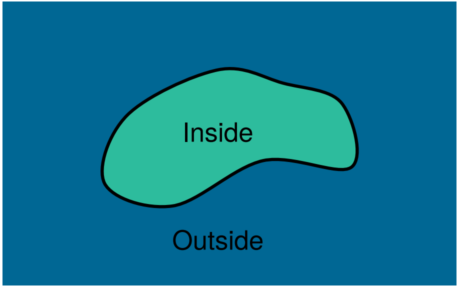
Closed, not simple.



Not closed, not simple.

Jordan curve theorem – /ʒɔʀdɑ̃/

If $\gamma : [a, b] \to \mathbb{C}$ is a simple closed curve then $\mathbb{C} \setminus \gamma([a, b])$ consists of two disjoints open connected sets, one bounded (*the inside*) and one unbounded (*the outside*).



This result seems *obvious*, nonetheless it is quite difficult to prove (note that we only assume that γ is continuous, so the curve can be quite nasty, e.g. Koch snowflake).

- We say that a curve $\gamma : [a,b] \to \mathbb{C}$ is **smooth** if γ is $\mathscr{C}^1$.

  Here γ is a function of a real variable, so by $\mathscr{C}^1$, I mean that $\Re(\gamma)$ and $\Im(\gamma)$ are $\mathscr{C}^1$ as real functions.

- We say that a curve $\gamma : [a,b] \to \mathbb{C}$ is **piecewise-smooth** if there exist $t_0, \dots, t_n \in [a,b]$ such that $a = t_0 < t_1 < \cdots < t_n = b$ and γ is $\mathscr{C}^1$ on (t_k, t_{k+1}) for $k = 0, \dots, n-1$.

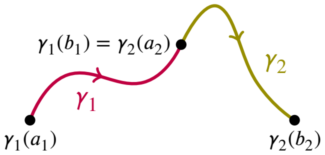
Figure: A piecewise-smooth curve.

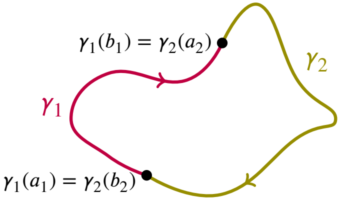
Definition: concatenation/sum

Given two curves $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ such that $\gamma_1(b_1) = \gamma_2(a_2)$, we define the **concatenation** (or **sum**) of γ_1 and γ_2 by $(\gamma_1 + \gamma_2) : [a_1, b_1 + b_2 - a_2] \to \mathbb{C}$ where

$$(\gamma_1 + \gamma_2)(t) := \begin{cases} \gamma_1(t) & \text{if } t \in [a_1, b_1] \\ \gamma_2(t + a_2 - b_1) & \text{if } t \in [b_1, b_1 + b_2 - a_2] \end{cases}$$

It is obviously a curve, i.e. it is continuous.





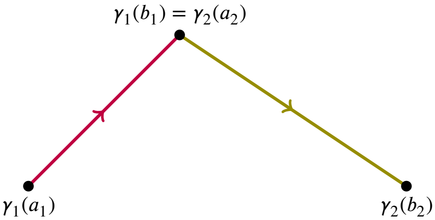
Curves – 6

Proposition

If γ_1 and γ_2 are piecewise-smooth then so is $(\gamma_1 + \gamma_2)$.

CAREFUL

It is possible for γ_1 and γ_2 to be smooth but for $(\gamma_1 + \gamma_2)$ not to be smooth, but only piecewise-smooth, since it may not be smooth at the *gluing point*.

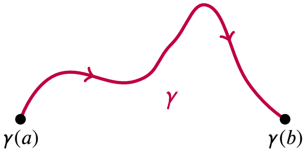


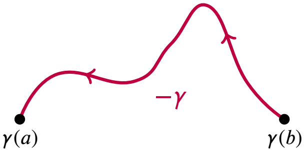
Curves are naturally oriented by the usual orientation of $[a,b]$: γ goes from $\gamma(a)$ to $\gamma(b)$.

Definition: reversed curve

For a curve $\gamma : [a,b] \to \mathbb{C}$ we define $(-\gamma) : [a,b] \to \mathbb{C}$ by $(-\gamma)(t) := \gamma(a+b-t)$.

It is the same curve but with reversed orientation: it is traversed from $\gamma(b)$ to $\gamma(a)$.





Curves – 8

Definition: positive orientation (closed simple curves)

We say that a simple closed curve γ is **positively oriented** if γ keeps its inside on its left.

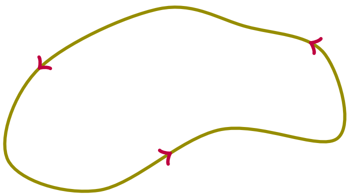


Figure: A positively oriented simple closed curve

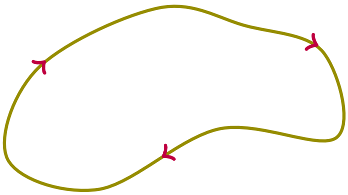
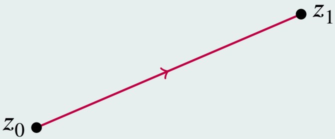


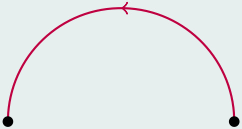
Figure: A simple closed curve which is NOT positively oriented

Examples

- $\gamma : [0,1] \to \mathbb{C}$, $\gamma(t) = (1-t)z_0 + tz_1$.



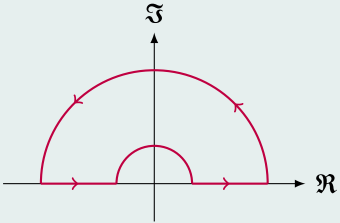
- $\gamma : [0,1] \to \mathbb{C}$, $\gamma(t) = e^{i\pi t}$.



Example

Pick $\varepsilon \in (0,1)$ and set $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ where

- $\gamma_1 : [0,\pi] \to \mathbb{C}$, $\gamma_1(t) = e^{it}$.
- $\gamma_2 : [-1,-\varepsilon] \to \mathbb{C}$, $\gamma_2(t) = t$.
- $\gamma_3 : [0,\pi] \to \mathbb{C}$, $\gamma_3(t) = \varepsilon e^{i(\pi-t)}$.
- $\gamma_4 : [\varepsilon,1] \to \mathbb{C}$, $\gamma_4(t) = t$.



Integrals

Definition

For $f : [a, b] \to \mathbb{C}$ continuous we set

$$\int_a^b f(t)\mathrm{d}t := \int_a^b \Re(f(t))\mathrm{d}t + i\int_a^b \Im(f(t))\mathrm{d}t$$

Propositions

For $f : [a, b] \to \mathbb{C}$ continuous,

$$\Re\left(\int_a^b f(t)\mathrm{d}t\right) = \int_a^b \Re(f(t))\mathrm{d}t$$

$$\Im\left(\int_a^b f(t)\mathrm{d}t\right) = \int_a^b \Im(f(t))\mathrm{d}t$$

Definition: line integral

Let $S \subset \mathbb{C}$ and $f : S \to \mathbb{C}$ be continuous. Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise-smooth curve such that $\gamma([a, b]) \subset S$. Pick a subdivision $a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b]$ such that γ is C^1 on (t_k, t_{k+1}) then the **line integral of f along γ** is defined by

$$\int_\gamma f(z)\mathrm{d}z := \sum_{k=0}^{k-1} \int_{t_k}^{t_{k+1}} f(\gamma(t))\gamma'(t)\mathrm{d}t$$

It doesn't depend on the choice of the subdivision: any subdivision as above gives the same value for $\int_\gamma f$.

If γ is smooth, we can simply write

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t$$

Line integrals – 2

Proposition

- $\displaystyle\int_{-\gamma} f = -\int_\gamma f$
- $\displaystyle\int_{\gamma_1+\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$
- $\displaystyle\int_\gamma \alpha f + \beta g = \alpha \int_\gamma f + \beta \int_\gamma g, \quad \alpha, \beta \in \mathbb{C}$

Homework

Prove the above properties.

Line integrals – 3

Theorem: reparametrization

Let $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ be two curves.
Assume that $\gamma_1 = \gamma_2 \circ \varphi$ where $\varphi : [a_1, b_1] \to [a_2, b_2]$ is a $\mathscr{C}^1$-diffeomorphism, then

- If φ is increasing then $\displaystyle\int_{\gamma_1} f = \int_{\gamma_2} f$, *and we say that φ preserves the orientation.*
- If φ is decreasing then $\displaystyle\int_{\gamma_1} f = -\int_{\gamma_2} f$, *and we say that φ reverses the orientation.*

Definition: length (or arclength) of a curve

The **(arc)length** of a piecewise smooth[1] curve $\gamma : [a, b] \to \mathbb{C}$ is $\operatorname{length}(\gamma) := \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left|\gamma'(t)\right| \mathrm{d}t$.

Proposition

If $\gamma : [a, b] \to \mathbb{C}$ is a smooth curve then $\operatorname{length}(\gamma) = \int_a^b \left|\gamma'(t)\right| \mathrm{d}t$.

Proposition

Let $S \subset \mathbb{C}$, $f : S \to \mathbb{C}$ continuous. Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise-smooth curve such that $\gamma([a, b]) \subset S$ then

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \leq \left(\max_{\gamma([a,b])} |f|\right) \operatorname{length}(\gamma)$$

The max is achieved since $\gamma([a, b])$ is compact and $|f|$ continuous.

[1] A continuous curve may not be rectifiable, e.g. Koch snowflake.

The real Green's theorem (from multivariable calculus) – 1

Green's theorem (real version)

Let $S \subset \mathbb{R}^2$ be a regular region with a piecewise smooth boundary ∂S which is assumed to be positively oriented.
Let $\mathbf{F} : U \to \mathbb{R}^2$ be a $\mathscr{C}^1$-vector field where $U \subset \mathbb{R}^2$ is open and $S \subset U$.
Then

$$\int_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}x\mathrm{d}y$$

or using the other notation for the line integral of a vector field

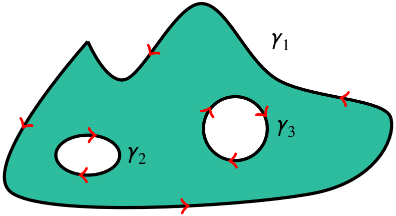
$$\int_{\partial S} P\mathrm{d}x + Q\mathrm{d}y = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\mathrm{d}y$$

I suggest you to review the details of Green's theorem from your multivariable calculus course.
See for instance:
`http://www.math.toronto.edu/campesat/ens/1920/0310-notes.pdf`

The real Green's theorem (from multivariable calculus) – 2

- S regular region means that S is bounded and that $\overline{\mathring{S}} = S$,
  or equivalently that S is bounded and that $\forall u \in \partial S,\ \forall r > 0,\ D_r(u) \cap \mathring{S} \neq \emptyset$.

- ∂S positively oriented means that each piece of ∂S is parametrized by a closed simple curve such that the interior $\mathring{S}$ is on its left.



- If $\partial S = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_\ell$ then $\displaystyle\int_{\partial S} = \int_{\gamma_1} + \cdots + \int_{\gamma_\ell}$.

The complex Green's theorem – version 1

Green's theorem (complex version) - 1

Let $S \subset \mathbb{C}$ be a regular region with a piecewise smooth boundary ∂S which is assumed to be positively oriented.
Let $U \subset \mathbb{C}$ open such that $S \subset U$.
Let $f : U \to \mathbb{C}$ be such that $(x, y) \mapsto (\Re(f), \Im(f))$ is $\mathscr{C}^1$ in the real sense (i.e. as a real multivariable function).
Then,

$$\int_{\partial S} f(z)\mathrm{d}z = i \iint_S \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) \mathrm{d}x\mathrm{d}y$$

where

$$\frac{\partial f}{\partial x} := \frac{\partial \Re(f)}{\partial x} + i \frac{\partial \Im(f)}{\partial x}$$

and

$$\frac{\partial f}{\partial y} := \frac{\partial \Re(f)}{\partial y} + i \frac{\partial \Im(f)}{\partial y}$$

Homework: check it!

The complex Green's theorem – version 2

Green's theorem (complex version) - 2

Let $S \subset \mathbb{C}$ be a regular region with a piecewise smooth boundary ∂S which is assumed to be positively oriented.
Let $U \subset \mathbb{C}$ open such that $S \subset U$.
Let $f : U \to \mathbb{C}$ be such that $(x, y) \mapsto (\Re(f), \Im(f))$ is $\mathscr{C}^1$ in the real sense (i.e. as a real multivariable function).
Then,

$$\int_{\partial S} f(z)\mathrm{d}z = 2i \iint_S \frac{\partial f}{\partial \bar{z}} \,\mathrm{d}x\mathrm{d}y$$

where

$$\frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

and

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

Extra: Koch snowflake,
a continuous nowhere differentiable simple closed curve