Usual complex functions – 1

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The complex exponential function – 1

Definition: the complex exponential function

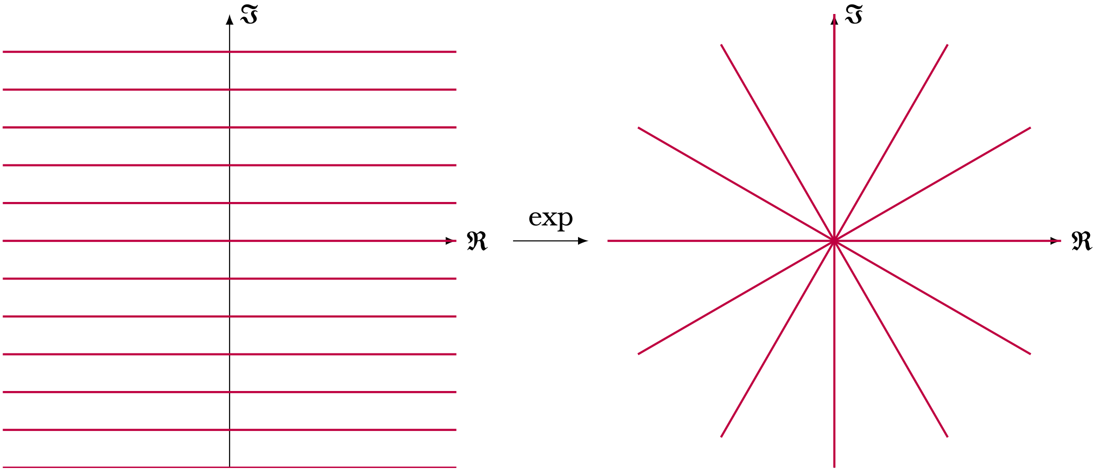
For $z = x + iy \in \mathbb{C}$, we set $e^z := e^x e^{iy}$, it defines the *complex exponential function*

$$\exp : \begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} \\ z & \mapsto & e^z \end{array}$$

Here $x, y \in \mathbb{R}$, so e^x is the usual real exponential and $e^{iy} := \cos(y) + i\sin(y)$ as defined last week.

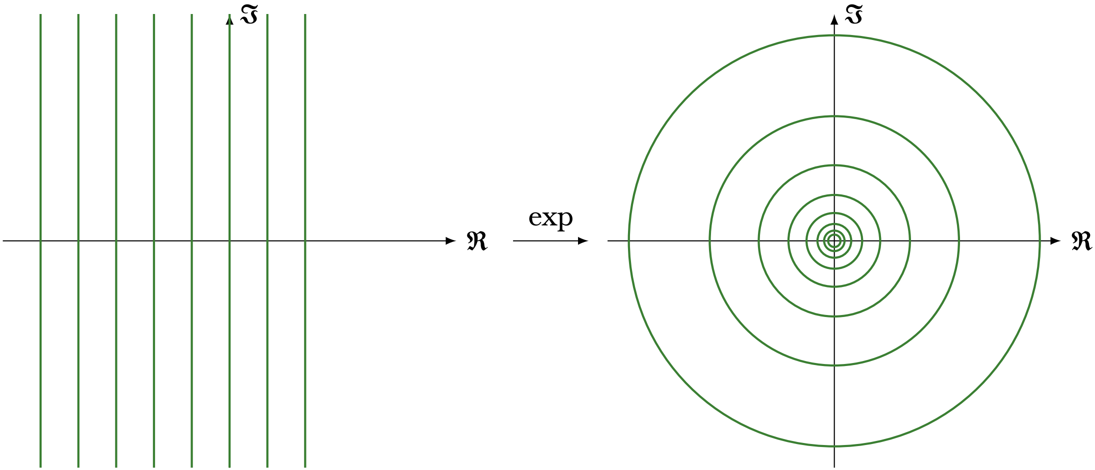
The complex exponential function – 2

The horizontal line $\Im(z) = c$ is mapped by the complex exponential to the open semiline $\{e^{x+ic} \ : \ x \in \mathbb{R}\} = \{\rho e^{ic} \ : \ \rho > 0\}$:

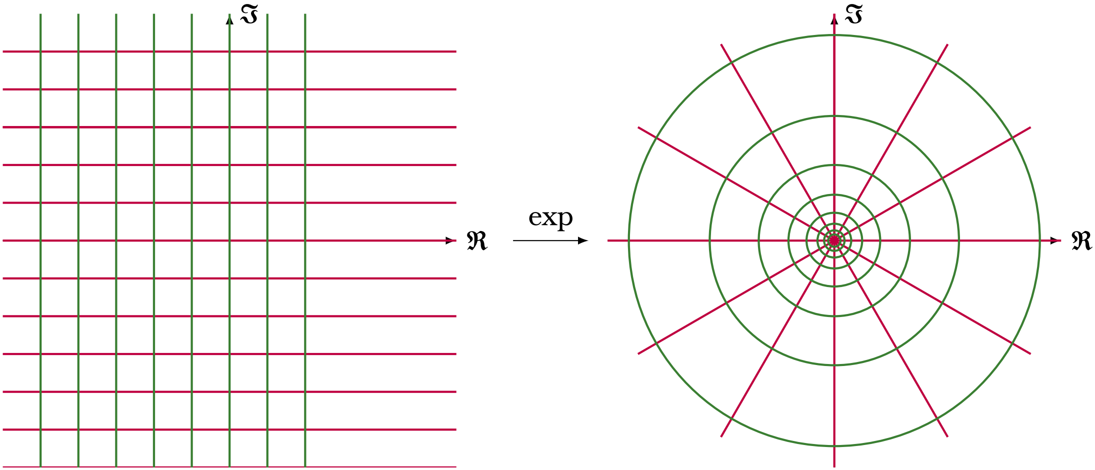


The complex exponential function – 3

The vertical line $\Re(z) = c$ is mapped by the complex exponential to the circle $\left\{e^{c+iy} \;:\; y \in \mathbb{R}\right\} = \mathscr{C}(0, e^c)$:



The complex exponential function – 4



Proposition

- $\mathfrak{R}(e^z) = e^{\mathfrak{R}(z)} \cos(\mathfrak{I}(z))$
- $\mathfrak{I}(e^z) = e^{\mathfrak{R}(z)} \sin(\mathfrak{I}(z))$
- $|e^z| = e^{\mathfrak{R}(z)}$
- $e^{z_1+z_2} = e^{z_1} e^{z_2}$
- exp is $2i\pi$-periodic: $e^{z+2i\pi} = e^z$
- $\mathrm{Range}(\exp) = \mathbb{C} \setminus \{0\}$
- $e^z = 1 \Leftrightarrow \exists n \in \mathbb{Z},\ z = 2i\pi n$

The complex logarithm function(s) – 1

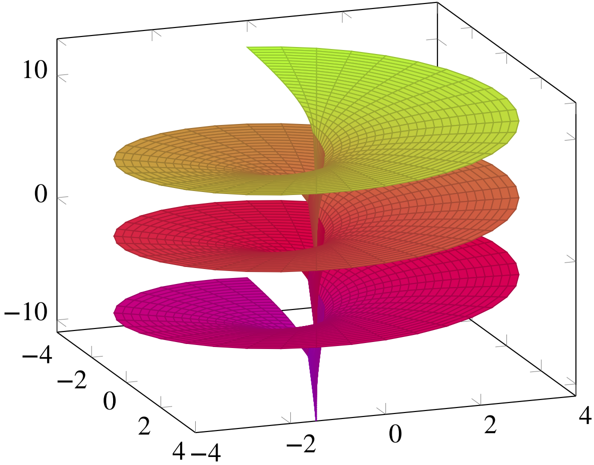
We want to define the complex logarithm as the inverse of the complex exponential, but it is not possible since the complex exponential is not injective (it is $2i\pi$-periodic).
Indeed, for $z \in \mathbb{C} \setminus \{0\}$, we have

$$e^w = z \Leftrightarrow w \equiv \log|z| + i\arg(z) \mod 2i\pi$$

So the function we obtain is *multivalued* since it is well-defined only up to $2i\pi$:

$$\begin{aligned} \log(z) &:= \log|z| + i\,\mathrm{Arg}(z) + 2i\pi\mathbb{Z} \\ &= \left\{\log|z| + i\,\mathrm{Arg}(z) + 2i\pi n \;:\; n \in \mathbb{Z}\right\} \end{aligned}$$

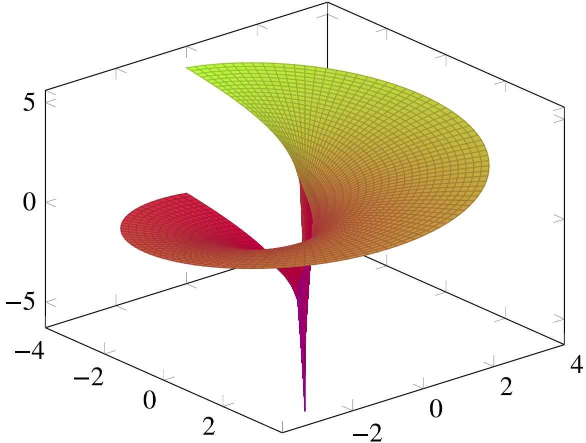
Below is the "graph" of the imaginary part of the multivalued logarithm $\log(z) := \log|z| + i\,\mathrm{Arg}(z) + 2i\pi\mathbb{Z}$.



The distance between two consecutive *branches* is 2π, it corresponds to the various values over one $z \in \mathbb{C} \setminus \{0\}$.

The complex logarithm function(s) – 2

Then to make $\exp$ one-to-one, we need to shrink its domain to a imaginary band of height 2π.
For $\alpha \in \mathbb{R}$, we set $\tilde{B}_\alpha \coloneqq \{z \in \mathbb{C} \ : \ \alpha \leq \Im(z) < \alpha + 2\pi\}$.
Then $\exp_{|\tilde{B}_\alpha} : \tilde{B}_\alpha \to \mathbb{C} \setminus \{0\}$ is a bijection and its inverse $\tilde{\varphi}_\alpha : \mathbb{C} \setminus \{0\} \to \tilde{B}_\alpha$ is well-defined.
Below is the graph of the imaginary part of such a φ_α:



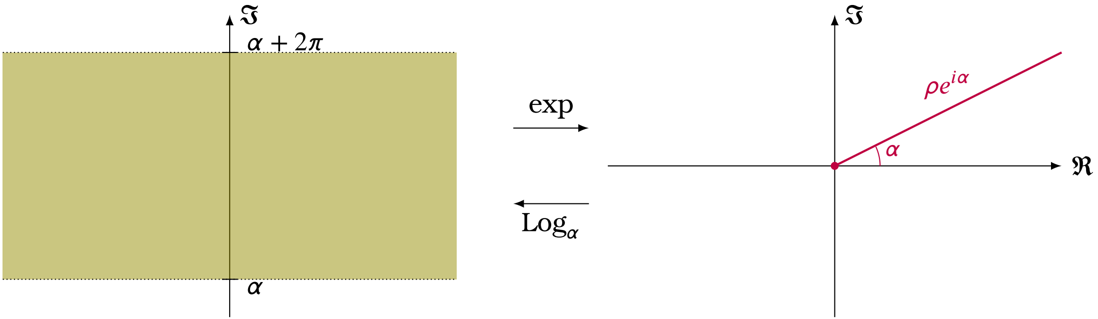
But, as we can see on the graph, it is not continuous: for $\varepsilon > 0$ small enough,
$\varphi_\alpha\left(e^{i(\alpha-\varepsilon)}\right) = \varphi_\alpha\left(e^{i(\alpha-\varepsilon+2\pi)}\right) = i(\alpha-\varepsilon+2\pi) \in \tilde{B}_\alpha$.

Hence
$\lim_{\varepsilon \to 0^+} \varphi_\alpha\left(e^{i(\alpha-\varepsilon)}\right) = i(\alpha + 2\pi) \neq i\alpha = \varphi_\alpha\left(e^{i\alpha}\right)$.

The complex logarithm function(s) – 3

In order to get a continuous function, we need to shrink again the domain of $\exp$.
For $\alpha \in \mathbb{R}$, we set $B_\alpha := \{z \in \mathbb{C} \ : \ \alpha < \Im(z) < \alpha + 2\pi\}$ then $\exp_{|B_\alpha} : B_\alpha \to \mathbb{C} \setminus \left\{\rho e^{i\alpha} \ : \ \rho \geq 0\right\}$ is a bijection. Note that we need to remove a closed semiline in the range!

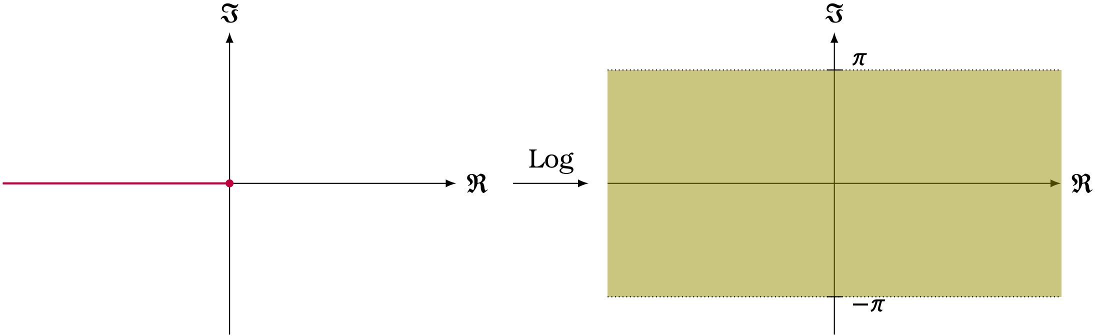
The inverse $\operatorname{Log}_\alpha : \mathbb{C} \setminus \left\{\rho e^{i\alpha} \ : \ \rho \geq 0\right\} \to B_\alpha$ is continuous, it is *the complex logarithm branch corresponding to the cut* α. Note that $\operatorname{Log}_\alpha(z) = \log|z| + i\operatorname{Arg}_\alpha(z)$ where $\operatorname{Arg}_\alpha(z) \in (\alpha, \alpha + 2\pi)$.



When α is not given, you may assume that $\alpha = -\pi$ (that's just a common convention), it is the *principal branch of the complex logarithm*:

$$\operatorname{Log} : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \to \{z \in \mathbb{C} \ : \ -\pi < \Im(z) < \pi\}$$

$$\operatorname{Log}(z) = \log|z| + i\operatorname{Arg}(z), \quad \text{where } \operatorname{Arg}(z) \in (-\pi, \pi)$$



The complex logarithm function(s) – 5

The notation is really ambiguous and depends on authors…

In the textbook:

- $\log$ is *multivalued/set-valued*: it is defined for $z \in \mathbb{C} \setminus \{0\}$ but is only well-defined modulo $2i\pi$.
- $\operatorname{Log} : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \to \{z \in \mathbb{C} \ : \ -\pi < \Im(z) < \pi\}$ is the principal branch of the complex logarithm, $\operatorname{Log}(z) = \log|z| + i \operatorname{Arg}(z)$ where $\operatorname{Arg}(z) \in (-\pi, \pi)$.

Note that

$$\forall z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}, \log(z) = \operatorname{Log}(z) + 2i\pi\mathbb{Z} = \left\{\operatorname{Log}(z) + 2i\pi n \ : \ n \in \mathbb{Z}\right\}$$

The complex logarithm function(s) – 6

Proposition

- $\log(z_1 z_2) \equiv \log(z_1) + \log(z_2) \mod 2i\pi$
- $\log\left(\frac{z_1}{z_2}\right) \equiv \log(z_1) - \log(z_2) \mod 2i\pi$

Here we have equalities as ***sets/multivalued functions/modulo*** $2i\pi$.

BEWARE

For a given branch of the complex logarithm, in general $\mathrm{Log}_\alpha(z_1 z_2) \neq \mathrm{Log}_\alpha(z_1) + \mathrm{Log}_\alpha(z_2)$.

In the following example, I work with the principal branch of the complex logarithm ($\alpha = -\pi$) then:
Let $z_1 = z_2 = e^{2i\frac{\pi}{3}}$ then

$$\mathrm{Log}(z_1 z_2) = \mathrm{Log}\left(e^{4i\frac{\pi}{3}}\right) = \mathrm{Log}\left(e^{-2i\frac{\pi}{3}}\right) = -2i\frac{\pi}{3} \neq 4i\frac{\pi}{3} = 2i\frac{\pi}{3} + 2i\frac{\pi}{3} = \mathrm{Log}(z_1) + \mathrm{Log}(z_2)$$