Taylor's theorem

The one-variable case (From MAT137)

Def: $I \subset \mathbb{R}$ open interval, $f : I \longrightarrow \mathbb{R}$, $a \in I$.
assume that f is k-th time differentiable at a, then
the k-th order Taylor polynomial of f at a is

$$P_{a,k}(x) = f(a) + f'(a)\,x + \frac{f''(a)}{2}x^2 + \dots + \frac{f^{(k)}(a)}{k!}x^k$$

$$= \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!} x^j$$

Prop: $P_{a,k}$ is the unique polynomial of degree at most k s.t.

$$P_{a,k}(0) = f(a),\ P_{a,k}'(0) = f'(a),\ P_{a,k}''(0) = f''(a), \dots, P_{a,k}^{(k)}(0) = f^{(k)}(a)$$

Theorem: (Taylor or Taylor-Young)

$I \subset \mathbb{R}$ ~~open~~ interval, $f : I \longrightarrow \mathbb{R}$ of class C^{k-1} on I, $a \in I$

If $f^{(k)}(a)$ exists then $\leftarrow$ I don't assume that $f^{(k)}$ is C^0 at a

Then $f(a+h) = P_{a,k}(h) + E(h)$ where $\dfrac{E(h)}{h^k} \xrightarrow[h \to 0]{} 0$

Δ We set $E(h) = f(a+h) - P_{a,k}(h)$ and $G(h) = h^k$

By L'Hopital's rule applied $(k-1)$ times (check the assumptions for each)

$$\lim_{h\to 0} \frac{E(h)}{G(h)} \overset{L'H}{=} \lim_{h\to 0} \frac{E'(h)}{G'(h)} \overset{L'H}{=} \dots \overset{L'H}{=} \lim_{h\to 0} \frac{E^{(k-1)}(h)}{G^{(k-1)}(h)}$$

(we can go up to there since f is C^{k-1})

by definition of $P_{a,k}$ $\leftarrow$ $$= \lim_{h\to 0} \frac{f^{(k-1)}(a+h) - f^{(k-1)}(a) - h f^{(k)}(a)}{k!\,h}$$

since $f^{(k)}(a)$ exists, it is the first term of the lim $\leftarrow$ $$= \lim_{h\to 0} \frac{f^{(k-1)}(a+h) - f^{(k-1)}(a)}{h} \cdot \frac{1}{k!} - \frac{f^{(k)}(a)}{k!} = 0$$

$\square$

Theorem (Taylor-Lagrange) particularly f is C^k on I

$I \subset \mathbb{R}$ interval, $f: I \to \mathbb{R}$ $(k+1)$-times differentiable on I, $a \in I$

Let $h \in \mathbb{R} \setminus \{0\}$ s.t. $\begin{cases} [a, a+h] \subset I \text{ if } h > 0 \\ \text{or} \\ [a+h, a] \subset I \text{ if } h < 0 \end{cases}$

Then $\begin{cases} \exists \xi \in (a, a+h) \text{ if } h > 0 \\ \text{or} \\ \exists \xi \in (a+h, a) \text{ if } h < 0 \end{cases}$ s.t.

$$f(a+h) = P_{a,k}(h) + \frac{f^{(k+1)}(\xi)}{(k+1)!} h^{k+1} = \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!} h^j + \frac{f^{(k+1)}(\xi)}{(k+1)!} h^{k+1}$$

Δ WLOG, we may assume that $h > 0$

Define $\varphi : [a, a+h] \to \mathbb{R}$ by

$$\varphi(t) = f(a+h) - f(t) - f'(t)(a+h-t) - \dots - \frac{f^{(k)}(t)}{k!}(a+h-t)^k - \frac{A}{(k+1)!}(a+h-t)^{k+1}$$

where we pick $A \in \mathbb{R}$ s.t. $\varphi(a) = 0$, notice that $\varphi(a+h) = 0$

Since φ is C^0 on $[a, a+h]$ and differentiable on $(a, a+h)$,

by Rolle's theorem, $\exists \xi \in (a, a+h)$ s.t. $\varphi'(\xi) = 0$

But, $\forall t \in (a, a+h)$, $\varphi'(t) = -\frac{f^{(k+1)}(t)}{k!}(a+h-t)^k + \frac{A}{k!}(a+h-t)^k$

(when we compute the derivative, the other terms cancel)

Hence $$0 = \varphi'(\xi) = -\frac{f^{(k+1)}(\xi)}{k!}\underbrace{(a+h-\xi)^k}_{\neq 0} + \frac{A}{k!}\underbrace{(a+h-\xi)^k}_{\neq 0}$$

$$\Rightarrow A = f^{(k+1)}(\xi)$$

Then $0 = \varphi(a) = f(a+h) - P_{a,k}(h) - \frac{f^{(k+1)}(\xi)}{(k+1)!} h^{k+1}$ $\square$

Taylor's theorem in several variables

At order 1

Prop: $\underset{\text{open}}{U \subset} \mathbb{R}^m$, $f : U \longrightarrow \mathbb{R}$, $a \in U$

If f is differentiable at a then

$$f(a+h) = f(a) + \sum_{j=1}^{m} \frac{\partial f}{\partial x_j}(a) h_j + E(h)$$

where $\frac{E(h)}{\|h\|} \xrightarrow[h \to 0]{} 0$

Δ It is just the definition noticing that $d_a f(h) = \sum_{j=1}^{m} \frac{\partial f}{\partial x_j}(a) h_j$ $\square$

At order 2

⚠

Theorem: $U \subset \mathbb{R}^m$ open, $f : U \longrightarrow \mathbb{R}$ of class C^2, $a \in U$, $h \in \mathbb{R}^m$

Assume that $\forall t \in [0,1]$, $a+th \in U$

Then $\exists \theta \in (0,1)$ s.t. $f(a+h) = f(a) + \sum_{j=1}^{m} \frac{\partial f}{\partial x_j}(a) h_j + \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial x_i \partial x_j}(a+\theta h) h_i h_j$

Δ Define $\varphi : [0,1] \to \mathbb{R}$ by $\varphi(t) = f(a+th)$

By the chain-rule: $\forall t \in (0,1)$, $\varphi'(t) = \sum_{j=1}^{m} \frac{\partial f}{\partial x_j}(a+th) h_j$

$$\varphi''(t) = \sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial x_i \partial x_j}(a+th) h_i h_j$$

Then by the one variable Taylor-Lagrange $\exists \theta \in (0,1)$

s.t. $\varphi(1) = \varphi(0) + \varphi'(0) + \frac{1}{2} \varphi''(\theta)$

$\square$

Theorem. *Let $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}$ of class C^2, $\mathbf{a} \in U$.*
Then

$$f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{a})h_i + \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{a})h_i h_j + E(\mathbf{h})$$

where $\lim_{\mathbf{h}\to 0} \frac{E(\mathbf{h})}{\|\mathbf{h}\|^2} = 0$.

Proof. Let $\mathbf{h} \in \mathbb{R}^n$ be of norm small enough to ensure that $\forall t \in [0,1]$, $\mathbf{a} + t\mathbf{h} \in U$.
By the previous theorem, there exists $\theta \in (0,1)$ such that

$$f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{a})h_i + \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{a}+\theta\mathbf{h})h_i h_j$$

Hence

$$\begin{aligned} E(\mathbf{h}) &= f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{a})h_i - \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{a})h_i h_j \\ &= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} \left(\frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{a}+\theta\mathbf{h}) - \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{a}) \right) h_i h_j \end{aligned}$$

So that

$$\frac{E(\mathbf{h})}{\|\mathbf{h}\|^2} = \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} \left(\frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{a}+\theta\mathbf{h}) - \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{a}) \right) \frac{h_i h_j}{\|\mathbf{h}\|^2}$$

Notice that by continuity of the second order partial derivatives

$$\lim_{\mathbf{h}\to 0} \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{a}+\theta\mathbf{h}) - \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{a}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{a}) - \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{a}) = 0$$

and that $\frac{|h_i h_j|}{\|\mathbf{h}\|^2} = \frac{|h_i|}{\|\mathbf{h}\|}\frac{|h_j|}{\|\mathbf{h}\|} \leq 1$.
Hence $\lim_{\mathbf{h}\to 0} \frac{E(\mathbf{h})}{\|\mathbf{h}\|^2} = 0$. ∎

Definition: $\mathcal{U} \subset \mathbb{R}^m$ open, $f : \mathcal{U} \longrightarrow \mathbb{R}$, $a \in \mathcal{U}$

We define the Hessian matrix of f at a by

$$\mathcal{H}_f(a) := \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(a) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m}(a) \\ \vdots & & \\ \frac{\partial^2 f}{\partial x_m \partial x_1}(a) & \cdots & \frac{\partial^2 f}{\partial x_m \partial x_m}(a) \end{pmatrix} \in M_{m,m}(\mathbb{R})$$

whenever it makes sense.

Remark: $$\sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)\, h_i h_j = h^t \cdot \mathcal{H}_f(a) \cdot h$$

At higher-order

Theorem: $\mathcal{U} \subset \mathbb{R}^m$ open, $f : \mathcal{U} \longrightarrow \mathbb{R}$ of class C^k, $a \in \mathcal{U}$

$$f(a+h) = \sum_{|\alpha| \leq k} \frac{\partial^\alpha f(a)}{\alpha!} h^\alpha + E(h), \quad \frac{E(h)}{\|h\|^k} \xrightarrow[h \to 0]{} 0$$

where $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m_{\geq 0}$

$|\alpha| = \alpha_1 + \dots + \alpha_m$

$\alpha! = \alpha_1! \alpha_2! \cdots \alpha_m!$

$h^\alpha = h_1^{\alpha_1} h_2^{\alpha_2} \cdots h_m^{\alpha_m}$

$\partial^\alpha f(a) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}}(a)$

Δ We admit this one $\square$

How to compute some multivariable Taylor polynomials.

Ex: $\dfrac{e^{x-2y}}{1+x^2-y} = \dfrac{e^{x-2y}}{1-(y-x^2)}$

$$= \left(1+(x-2y)+\frac{(x-2y)^2}{2}+\dots\right)\left(1+(y-x^2)+(y-x^2)^2+\dots\right)$$

$$= 1+y-x^2+(y-x^2)^2+(x-2y)+(x-2y)(y-x^2)+\frac{(x-2y)^2}{2}+\dots$$

$$= \underbrace{1}_{\text{order 0}}+\underbrace{x-y}_{\text{order 1}}\underbrace{-\frac{x^2}{2}-xy+y^2}_{\text{order 2}}+E(x,y)$$

where $\dfrac{E(x,y)}{\|(x,y)\|^2} \xrightarrow[(x,y)\to(0,0)]{} 0$

Hence $P_{(0,0),2}(x,y) = 1+x-y-\frac{x^2}{2}-xy+y^2$

Homework: Questions from §2.6 of the online lecture notes "Basic Skill: 1→4"

Do not attempt the "advanced": I gave alternative proofs in class

Higher order partial derivative : polar coordinates

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ of class C^2

Define $S = \{(r,\theta) \in \mathbb{R}^2,\ r > 0\}$ and $\varphi : S \to \mathbb{R}$ by

$$\varphi(r,\theta) = f(r\cos\theta, r\sin\theta)$$

From Oct 22 :

$$\partial_r \varphi = \cos\theta\, \partial_x f + \sin\theta\, \partial_y f$$

$$\partial_\theta \varphi = -r\sin\theta\, \partial_x f + r\cos\theta\, \partial_y f$$

Comment : by $\partial_r \varphi$, I mean $\frac{\partial \varphi}{\partial r}(r,\theta)$ and by $\partial_x f$, I mean $\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)$ } to lighten the notation

Hence :

$$\partial_r^2 \varphi = \cos^2\theta\, \partial_x^2 f + \cos\theta\sin\theta\, \partial_y\partial_x f + \cos\theta\sin\theta\, \partial_x\partial_y f + \sin^2\theta\, \partial_y^2 f$$

Clairaut's thm $\leftarrow$ $= \cos^2\theta\, \partial_x^2 f + 2\cos\theta\sin\theta\, \partial_x\partial_y f + \sin^2\theta\, \partial_y^2 f$

$\partial_r \varphi$

$$\partial_\theta^2 \varphi = -r\cos\theta\, \partial_x f + r^2\sin^2\theta\, \partial_x^2 f - r^2\sin\theta\cos\theta\, \partial_y\partial_x f$$
$$- r\sin\theta\, \partial_y f + r^2\cos^2\theta\, \partial_y^2 f - r^2\sin\theta\cos\theta\, \partial_x\partial_y f$$

Clairaut's thm $= -r\partial_r \varphi + r^2\sin^2\theta\, \partial_x^2 f + r^2\cos^2\theta\, \partial_y^2 f - 2r^2\sin\theta\cos\theta\, \partial_x\partial_y f$

$$\underbrace{\Delta f := \partial_x^2 f + \partial_y^2 f}_{\text{Laplacian operator : heat eqn, wave eqn, ...}} = \partial_r^2 \varphi + \frac{1}{r}\partial_r \varphi + \frac{1}{r^2}\partial_\theta^2 \varphi$$

$$\partial_\theta \partial_r \varphi = -\sin\theta\, \partial_x f - r\cos\theta\sin\theta\, \partial_x^2 f + r\cos^2\theta\, \partial_y \partial_x f$$
$$+ \cos\theta\, \partial_y f - r\sin^2\theta\, \partial_x \partial_y f + r\cos\theta\sin\theta\, \partial_y^2 f$$

$$= \frac{1}{r}\partial_\theta \varphi - \frac{r}{2}\sin(2\theta)\,\partial_x^2 f + \frac{r}{2}\sin(2\theta)\,\partial_y^2 f + r\cos(2\theta)\,\partial_x\partial_y f$$

Clairaut's thm & $\sin(2\theta) = 2\sin\theta\cos\theta$ & $\cos^2\theta - \sin^2\theta = \cos 2\theta$

Solving the one-dimensional wave equation (Extracurricular)

$$f : \begin{array}{l} \mathbb{R}^2 \longrightarrow \mathbb{R} \\ (x,t) \mapsto f(x,t) \end{array} \quad , \quad c > 0 \ , \ f \text{ of class } C^2$$

the eqn : $$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0$$

Define $\varphi : \mathbb{R}^2 \to \mathbb{R}$ by $\varphi(u,v) = f\left(\frac{u+v}{2}, \frac{u-v}{2c}\right)$

$$\frac{\partial \varphi}{\partial u} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2c}\frac{\partial f}{\partial t}$$

$$\frac{\partial^2 \varphi}{\partial v \partial u} = \frac{1}{4}\frac{\partial^2 f}{\partial x^2} - \frac{1}{4c}\frac{\partial^2 f}{\partial t \partial x} + \frac{1}{4c}\frac{\partial^2 f}{\partial x \partial t} - \frac{1}{4c^2}\frac{\partial^2 f}{\partial t^2}$$

$$= \frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}\right)$$

CCL : $$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0 \iff \frac{\partial^2 \varphi}{\partial u \partial v} = 0$$

$$\iff \varphi(u,v) = A(u) + B(v)$$

$A, B : \mathbb{R} \to \mathbb{R}$ of class C^2

$$\begin{cases} x = \frac{u+v}{2} \\ t = \frac{u-v}{2c} \end{cases} \iff \begin{cases} u = x + ct \\ v = x - ct \end{cases}$$

$$\underline{f(x,y) = A(x+ct) + B(x-ct) \ , \ A, B : \mathbb{R} \to \mathbb{R} \text{ of class } C^2}$$