Q2 : ① Let $p=(x_0,y_0)\in A\times B$

Since $x_0\in A$ open, there exists $\varepsilon_1>0$ st. $B(x_0,\varepsilon_1)\subset A$

Since $y_0\in B$ open, there exists $\varepsilon_2>0$ s.t. $B(y_0,\varepsilon_2)\subset B$

Let $\varepsilon=\min(\varepsilon_1,\varepsilon_2)$

Claim : $B(p,\varepsilon)\subset A\times B$

Indeed, let $q=(x,y)\in B(p,\varepsilon)$ then

$$\|x_0-x\|=\left(\sum_{i=1}^{m}(x_{0i}-x_i)^2\right)^{1/2}\leq\left(\sum_{i=1}^{m}(x_{0i}-x_i)^2+\sum_{j=1}^{m}(y_{0j}-y_j)^2\right)^{1/2}$$

$$=\|p-q\|$$

$$<\varepsilon\leq\varepsilon_1$$

hence $x\in B(x_0,\varepsilon_1)\subset A$

Similarly, $y\in B(y_0,\varepsilon_2)\subset B$

Hence $q=(x,y)\in A\times B$

We proved that $\forall p\in A\times B,\ \exists\varepsilon>0,\ B(p,\varepsilon)\subset A\times B$

so $A\times B$ is open

② $(A\times B)^c=\{(x,y):x\in A\text{ and }y\in B\}^c$

$=\{(x,y):x\notin A\text{ or }y\notin B\}$

$=(A^c\times\mathbb{R}^m)\cup(\mathbb{R}^n\times B^c)$

A^c is open, $\mathbb{R}^m$ too, hence $A^c\times\mathbb{R}^m$ is open by ①

Similarly $\mathbb{R}^n\times B^c$ is open

hence $(A\times B)^c$ is open as the union of 2 open sets

so $A\times B$ is closed as the complement of an open set

Q3. ① let $x \in \mathring{A}$, then there exists $\varepsilon > 0$ s.t. $B(x,\varepsilon) \subset A$

then $B(x,\varepsilon) \subset A \subset B$.

so we proved that $\exists \varepsilon > 0,\ B(x,\varepsilon) \subset B$, ie $x \in \mathring{B}$

Hence $x \in \mathring{A} \Rightarrow x \in \mathring{B}$, otherwise stated $\mathring{A} \subset \mathring{B}$

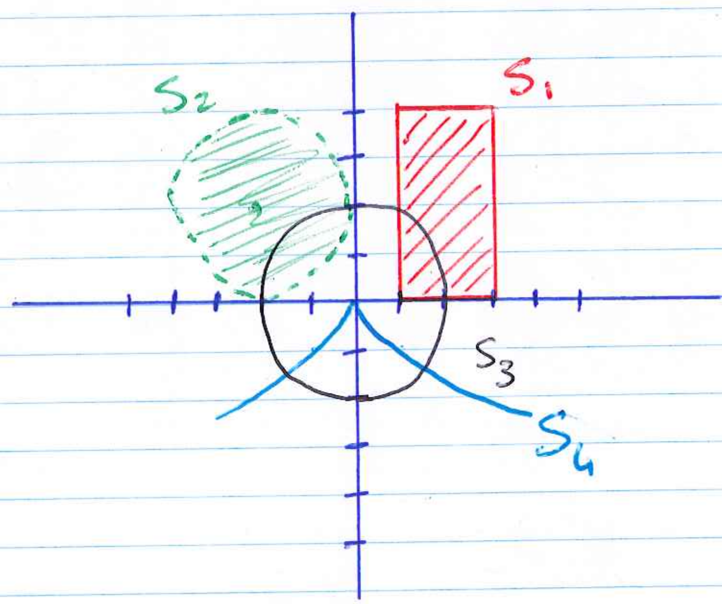
② let $x \in \bar{A}$.
let $\varepsilon > 0$, then $B(x,\varepsilon) \cap A \neq \phi$ since $x \in \bar{A}$

then $\phi \neq B(x,\varepsilon) \cap A \subset B(x,\varepsilon) \cap B$ so $B(x,\varepsilon) \cap B \neq \phi$

Hence $\forall \varepsilon > 0,\ B(x,\varepsilon) \cap B \neq \phi$, ie $x \in \bar{B}$

so $x \in \bar{A} \Rightarrow x \in \bar{B}$, ie $\bar{A} \subset \bar{B}$

Q4 ①



② • let $a, b \in S_2$, define $\gamma : [0,1] \longrightarrow \mathbb{R}^2$ by $\gamma(t) = (1-t)a + tb$

then γ is continuous, $\gamma(0) = a$, $\gamma(1) = b$

Notice that $S_2 = \{x \in \mathbb{R}^2 : \|x-p\| < 2\}$ where $p = (-2,2)$

and $\forall t \in [0,1]$, $\|\gamma(t) - p\| = \|(1-t)a + tb - p\|$

$= \|(1-t)(a-p) + t(b-p)\|$

$\leq (1-t)\|a-p\| + t\|b-p\|$

$< (1-t)2 + t2 = 2$

ie $\gamma(t) \in S_2$

- let $a, b \in S_1$  $a = (x_0, y_0)$, $b = (x_1, y_1)$

define $\gamma : [0,1] \longrightarrow \mathbb{R}^2$ by $\gamma(t) = (1-t)a + tb$

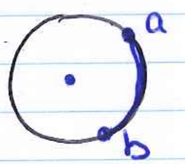
then $\gamma(0) = a$, $\gamma(1) = b$, γ is continuous

let $t \in [0,1]$, then $\gamma(t) = \big((1-t)x_0 + tx_1, (1-t)y_0 + ty_1\big)$

since $x_0, x_1 \in [1,3]$ interval, $(1-t)x_0 + tx_1 \in [1,3]$
$y_0, y_1 \in [0,4]$ interval, $(1-t)y_0 + ty_1 \in [0,4]$

hence $\gamma(t) \in [1,3] \times [0,4] = S_1$

Hence S_1 is path-connected

- let $a, b \in S_3$ 

then $a = 2(\cos\theta_1, \sin\theta_1)$
$b = 2(\cos\theta_2, \sin\theta_2)$

for some θ_1, θ_2

set $\gamma(t) = 2\Big(\cos\big((1-t)\theta_1 + t\theta_2\big), \sin\big((1-t)\theta_1 + t\theta_2\big)\Big)$

then $\gamma : [0,1] \longrightarrow \mathbb{R}^2$ is continuous, $\gamma(0) = a$, $\gamma(1) = b$

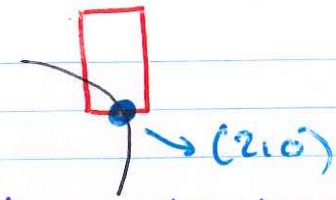
and $\forall t \in [0,1]$, $\gamma(t) \in S_3$

Hence S_3 is path-connected

- let $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ defined by $f(t) = (t^3, -t^2)$

then $S_4 = f(\mathbb{R})$ is path-connected as the image of $\mathbb{R}$ path-connected by f continuous

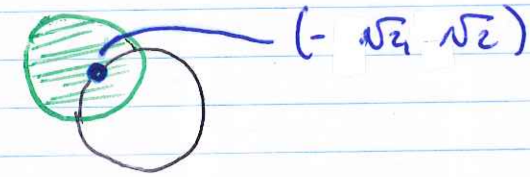
③ • $(2,0) \in S_1 \cap S_3$ so $S_1 \cap S_3 \neq \emptyset$



then $S_1 \cup S_3$ is path-connected as the union of two path-connected sets whose intersection is not $\emptyset$

• $2\left(\cos\left(\frac{3\pi}{4}\right), \sin\left(\frac{3\pi}{4}\right)\right) \in S_2 \cap S_3 \subset S_2 \cap (S_1 \cup S_3)$

$$= \left(-\sqrt{2}, \sqrt{2}\right)$$



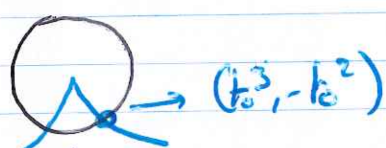
so $S_1 \cup S_2 \cup S_3 = S_2 \cup (S_1 \cup S_3)$ is path-connected as the union of two path-connected sets whose intersection is not empty

• let $f(t) = (t^3)^2 + (-t^2)^2 = t^6 + t^4$

then $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f(0) = 0$, $f(2) > 4$

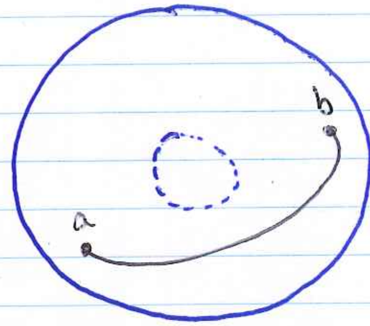
so by the IVT, $\exists\, t_0 \in (0,2)$ s.t. $(t_0^3)^2 + (-t_0^2)^2 = 4$

then $(t_0^3, -t_0^2) \in S_3 \cap S_4 \subset (S_1 \cup S_2 \cup S_3) \cap S_4 \neq \emptyset$



hence $S_1 \cup S_2 \cup S_3 \cup S_4$ is path-connected as the union of two path-connected sets with non-empty intersection

Q5.



let $a, b \in S$ then $a = r_1(\cos\theta_1, \sin\theta_2)$
$b = r_2(\cos\theta_2, \sin\theta_2)$

where $r_1, r_2 \in (1,2]$, $\theta_1, \theta_2 \in \mathbb{R}$

set $\gamma : [0,1] \longrightarrow \mathbb{R}$, $\gamma(t) = \Big((1-t)r_1 + t r_2\Big)\Big(\cos((1-t)\theta_1 + t\theta_2), \sin((1-t)\theta_1 + t\theta_2)\Big)$

then $\gamma(0) = a$, $\gamma(1) = b$ and γ is continuous

let $t \in [0,1]$, then $\|\gamma(t)\| = (1-t)r_1 + t r_2 \in (1,2]$

since $r_1, r_2 \in (1,2]$ interval.

hence $\forall t \in [0,1]$, $\gamma(t) \in S$.

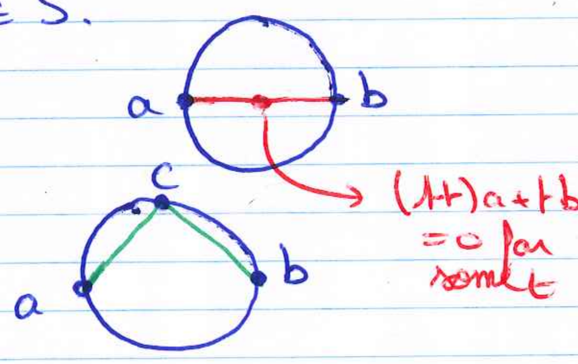
Hence S is path-connected

Q6 ① let $a, b \in S$
up to considering a third point c and finding a path from a to c and then from c to b, we may assume that a and b are not antipodal.

then set $\gamma(t) = \dfrac{(1-t)a + tb}{\|(1-t)a+tb\|}$ for $t \in [0,1]$

then $\gamma : [0,1] \to \mathbb{R}^n$ is well-defined since a, b are not antipodal, continuous, $\gamma(0) = a$, $\gamma(1) = b$, $\forall t \in [0,1]$, $\|\gamma(t)\| = 1$ ie $\gamma(t) \in S$.

Hence S is path-connected



② define $g : S \to \mathbb{R}$ by $g(x) = f(x) - f(-x)$

1st case: $\forall x \in S$, $g(x) = 0$ and then $\forall x \in S$, $f(x) = f(-x)$

2nd case: $\exists x_0 \in S$, $g(x_0) \neq 0$, let say $g(x_0) > 0$

then $g(-x_0) = f(-x_0) - f(x_0) = -g(x_0) < 0$

since S is path-connected and g continuous, by the IVT, $\exists x \in S$ s.t. $g(x) = 0$ ie $f(x) = f(-x)$

③ $h : \mathbb{R}^n \to \mathbb{R}$, $h(x) = \|x\| = \sqrt{\sum x_i^2}$ is continuous

hence $S = h^{-1}(\{1\})$ is closed as the inverse image of $\{1\}$ closed by h continuous

$\forall x \in S$, $\|x\| = 1$ so S is bounded
Hence S is compact as a closed and bounded set

④ by the EVT since S compact and $f : S \to \mathbb{R}$ continuous

Q7: • S_1 is closed by Q2 and not open since $S_1 \neq \mathbb{R}^2$ or $\emptyset$
S_1 is also bounded hence S_1 is compact

• $S_2 = f^{-1}((-\infty, 4))$ where $f(x,y) = (x+2)^2 + (y-2)^2$ continuous and $(-\infty, 4)$ open

hence S_2 is open (and not closed since $S_2 \neq \mathbb{R}^2$ or $\emptyset$)

S_2 is not compact since it is not closed but it is bounded

• $S_3 = g^{-1}(\{4\})$ where $g(x,y) = x^2+y^2$ continuous and $\{4\}$ closed
so S_3 is closed

S_3 is also bounded and hence compact

• S_4 is closed (why?) but not bounded

• S is not closed: $\forall k$, $(\frac{1}{k}, 1) \in S$ but
$\lim (1/k, 1) = (0,1) \notin S$
but $\in \bar{S}$ as a limit of a sequence of terms in S

hence $\bar{S} \setminus S \neq \emptyset$, ie $S \subsetneq \bar{S}$ and S not closed

S is not open: $(1-\frac{1}{k}, 0) \in S^c$ $\forall k$ but $\lim_{k \to +\infty} \notin S^c$

so S^c is not closed and S is not open.

~~Then~~ S is bounded
not compact

Q8. ① Actually there is no ε and no δ....

$\forall M>0, \exists K>0, \forall x\in\mathbb{R}^n, \|x\|>K \Rightarrow f(x)>M$

② Let $M=\max(f(0),1)>0$

then $\exists K>0$ s.t. $\|x\|>K \Rightarrow f(x)>M$ ⊛

$S=\overline{B}(0,K)$ is compact and f continuous hence f has a min on S

ie $\exists x_0\in S$ s.t. $\forall x\in\overline{B}(0,K),\ f(x)\geqslant f(x_0)$

Let's prove that $f(x_0)$ is a min on $\mathbb{R}^n$:

let $x\in\mathbb{R}^n$, if $x\in\overline{B}(0,K)$ then $f(x)\geqslant f(x_0)$

if $x\notin\overline{B}(0,K)$ then

$$f(x)\underset{\text{by ⊛}}{>} M \underset{\text{by def of } M}{\geqslant} f(0) \geqslant f(x_0)$$

$0\in\overline{B}(0,K)$ and $f(x_0)$ min on it

hence $\forall x\in\mathbb{R}^n,\ f(x)\geqslant f(x_0)$

Q9. ① Let $k \in \mathbb{N}_{\geq 0}$

Since S_k is not empty, $\exists\, x_k \in S_k \subset S_0$

We constructed a sequence $(x_k)_{k\geq 0}$ in S_0 compact

hence $\exists$ a subsequence $(x_{\varphi(j)})_{j\geq 0}$ convergent in S_0.

Since for j big enough $x_{\varphi(j)} \in S_k$ closed

we know that $\lim\limits_{j\to+\infty} x_{\varphi(j)} \in S_k$

hence $\lim\limits_{j\to+\infty} x_{\varphi(j)} \in \bigcap\limits_{k\geq 0} S_k \neq \emptyset$

② $\bigcap\limits_{k\geq 0} [k, +\infty) = \emptyset$: empty $\cap$ of decreasing closed sets

③ $\bigcap\limits_{k\geq 0} \left(1, 1 + \frac{1}{1+k}\right) = \emptyset$: ——— bounded sets

Q10 : 1) $u_{n+1} = u_n + \beta$ is an arithmetic sequence

hence $u_n = u_0 + n\beta$

2) a) $v_{n+1} = u_{n+1} - r$

$$= \alpha u_n + \beta - r$$

$$= \alpha(u_n - r) + \beta - r + \alpha r$$

$$= \alpha v_n + \underbrace{\beta - r + \alpha r}$$

$$= 0 \iff r = \frac{\beta}{1-\alpha}$$

Hence, for $r = \dfrac{\beta}{1-\alpha}$ $(\alpha \neq 1)$, $v_{n+1} = \alpha v_n$

b) then $v_n = v_0 \alpha^n$

ie, $u_n - r = (u_0 - r)\alpha^n$

$$\Rightarrow u_n = (u_0 - r)\alpha^n + r$$

$$u_n = \left(u_0 - \frac{\beta}{1-\alpha}\right)\alpha^n + \frac{\beta}{1-\alpha}$$

or

$$u_n = u_0 \alpha^n + \frac{\beta}{1-\alpha}\left(1 - \alpha^n\right)$$

Q11 : Write $u_n = (x_n, y_n)$

then $u_{n+1} = \frac{1}{2} u_n + (1,2)$

$\Rightarrow (x_{n+1}, y_{n+1}) = \frac{1}{2}(x_n, y_n) + (1,2) = \left(\frac{1}{2}x_n + 1, \frac{1}{2}y_n + 2\right)$

We are going to study (x_n) and then (y_n)

- $\begin{cases} x_0 = 1 \\ x_{n+1} = \frac{1}{2}x_n + 1 \end{cases}$

Hence $x_n = \left(1 - \frac{1}{1-\frac{1}{2}}\right)\left(\frac{1}{2}\right)^n + \frac{1}{1-\frac{1}{2}}$ by Q10

$= -\frac{1}{2^n} + 2 \xrightarrow[n \to +\infty]{} 2$

- $\begin{cases} y_0 = 0 \\ y_{n+1} = \frac{1}{2}y_n + 2 \end{cases}$

hence $y_n = \left(0 - \frac{2}{1-\frac{1}{2}}\right)\left(\frac{1}{2}\right)^n + \frac{2}{1-\frac{1}{2}}$

$= -\frac{4}{2^n} + 4 \xrightarrow[n \to +\infty]{} 4$

Hence $\lim_{n \to +\infty} u_n = (2,4)$

Q12 ① $\forall k,\ (0, \sqrt{2}/k) \in S$

but $\lim (0, \sqrt{2}/k) = (0,0) \in \bar{S} \setminus S$

hence $S \subsetneq \bar{S}$ and S is not closed

$\forall k,\ \left(\frac{\sqrt{2}}{k}, \sqrt{2}\right) \in S^c$

but $\lim \left(\frac{\sqrt{2}}{k}, \sqrt{2}\right) = (0, \sqrt{2}) \notin S^c$

hence S^c is not closed and S is not open

② • Let $(x_0, y_0) \in \mathbb{R}^2$, let $\varepsilon > 0$.

by the statement, $\exists r \in \mathbb{Q}^c$ s.t. $x_0 < r < x_0 + \varepsilon$

then $(r, y_0) \notin S$ but $(r, y_0) \in B((x_0, y_0), \varepsilon)$

ie: $\forall \varepsilon > 0,\ B((x_0,y_0),\varepsilon) \cap S^c \neq \emptyset$

hence $(x_0, y_0) \notin \mathring{S}$ ($\hookrightarrow$ means $B((x_0,y_0),\varepsilon) \not\subset S$)

Ccl: $\mathbb{R}^2 \subset (\mathring{S})^c \Rightarrow \mathring{S} = \emptyset$

• Let $(x_0, y_0) \in \mathbb{R}^2$, let $\varepsilon > 0$

$\exists q \in \mathbb{Q}$ s.t. $x_0 < q < x_0 + \frac{\varepsilon}{\sqrt{2}}$

$\exists r \in \mathbb{Q}^c$ s.t. $y_0 < r < y_0 + \frac{\varepsilon}{\sqrt{2}}$ check it!

then $(q, r) \in S$ and $(q, r) \in B((x_0,y_0), \varepsilon)$

ie $S \cap B((x_0,y_0),\varepsilon) \neq \emptyset$

Ccl: $\forall \varepsilon > 0,\ B((x_0,y_0),\varepsilon) \cap S \neq \emptyset$, ie $(x_0,y_0) \in \bar{S}$

So $\mathbb{R}^2 \subset \bar{S} \Rightarrow \bar{S} = \mathbb{R}^2$

• $\partial S = \bar{S} \setminus \mathring{S} = \mathbb{R}^2 \setminus \emptyset = \mathbb{R}^2$

Q13 ① $f'(x_0) = \lim_{t \to 0} \dfrac{f(x_0+t) - f(x_0)}{t}$ exists

② there is a linear function $d_{x_0}f : \mathbb{R} \to \mathbb{R}$ s.t.

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0) - d_{x_0}f(h)}{h} = 0$$

or equivalently

$$f(x_0+h) = f(x_0) + d_{x_0}f(h) + E(h)$$

where $\lim_{h \to 0} \dfrac{E(h)}{h} = 0$

③ $\Rightarrow$ set $d_{x_0}f(h) = f'(x_0)\,h$

then $\dfrac{f(x_0+h) - f(x_0) - d_x f(h)}{h} = \dfrac{f(x_0+h) - f(x_0)}{h} - f'(x_0)$

$$\xrightarrow[h \to 0]{} f'(x_0) - f'(x_0) = 0$$

$\Leftarrow$ $\dfrac{f(x_0+t) - f(x_0)}{t} = \dfrac{f(x_0+t) - f(x_0) - d_{x_0}f(t) + d_{x_0}f(t)}{t}$

$$= \frac{f(x_0+t) - f(x_0) - d_{x_0}f(t)}{t} + \frac{\cancel{t}\, d_{x_0}f(1)}{\cancel{t}}$$

$$\xrightarrow[t \to 0]{} d_{x_0}f(1)$$

④ then $f'(x_0) = d_{x_0}f(1)$ and $d_{x_0}f(h) = f'(x_0)\,h$

Q14: (1) The partial derivatives of f exist for all $(x,y) \in \mathbb{R}^2$ and

$$\frac{\partial f}{\partial x}(x,y) = e^{xy}(yx + y^2 + 1)$$

$$\frac{\partial f}{\partial y}(x,y) = e^{xy}(x^2 + yx + 1)$$

are continuous on $\mathbb{R}^2$

Hence f is differentiable on $\mathbb{R}^2$ and

$$\nabla f(x,y) = e^{xy}(yx + y^2 + 1,\ x^2 + yx + 1)$$

$$d_{(x,y)} f(h,k) = e^{xy}(yx + y^2 + 1)h + e^{xy}(x^2 + yx + 1)k$$

(2) The partial derivatives of f exist for all $(x,y,z) \in \mathbb{R}^3$ and

$$\frac{\partial f}{\partial x}(x,y,z) = y + z$$

$$\frac{\partial f}{\partial y}(x,y,z) = x + z$$

$$\frac{\partial f}{\partial z}(x,y,z) = x + y$$

are continuous on $\mathbb{R}^3$.

Hence f is differentiable on $\mathbb{R}^3$ and

$$\nabla f(x,y,z) = (y+z,\ x+z,\ x+y)$$

$$d_{(x,y,z)} f(h,k,l) = (y+z)h + (x+z)k + (x+y)l$$

Q15 : ① f is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$

let's see at $(0,0)$:

$$|f(x,y)| = \frac{|xy|}{x^2+y^2} |y| \leqslant \frac{|y|}{2} \xrightarrow[(x,y)\to(0,0)]{} 0$$

Hence $f(x,y) \xrightarrow[(x,y)\to(0,0)]{} 0 = f(0,0)$

Ccl: f is continuous on $\mathbb{R}^2$

② On $\mathbb{R}^2 \setminus \{(0,0)\}$ the partial derivatives of f exist and

$$\frac{\partial f}{\partial x}(x,y) = y^2 \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{2x^3y}{(x^2+y^2)^2}$$

At $(0,0)$: $\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = 0$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = 0$$

$$\frac{\partial f}{\partial x}(0,y) = \frac{y^4}{y^4} \xrightarrow[y\to 0]{} 1 \neq \frac{\partial f}{\partial x}(0,0)$$

so $\frac{\partial f}{\partial x}$ is not continuous at $(0,0)$

$$\frac{\partial f}{\partial y}(x,x) = \frac{2x^4}{4x^4} \xrightarrow[x\to 0]{} \frac{1}{2} \neq \frac{\partial f}{\partial y}(0,0)$$

so $\frac{\partial f}{\partial y}$ is not continuous at $(0,0)$

③ We can't conclude from ② that f is differentiable at $(0,0)$ since the partial derivatives are not continuous

Assume by contradiction that f is differentiable at $(0,0)$ then

$$\partial_{(1,1)} f(0,0) = d_{(0,0)} f(1,1) = d_{(0,0)} f(1,0) + d_{(0,0)} f(0,1)$$
$$= \partial_{(1,0)} f(0,0) + \partial_{(0,1)} f(0,0)$$
$$= \frac{\partial f}{\partial x}(0,0) + \frac{\partial f}{\partial y}(0,0)$$
$$= 0 + 0 = 0$$

but $\partial_{(1,1)} f(0,0) = \lim_{t \to 0} \frac{f(t,t)}{t} = \frac{1}{2}$ ✗

Hence f is not differentiable at $(0,0)$

$\times$ ——— $\times$

We may also directly do:

Assume ~~that~~ f is differentiable at $(0,0)$

then $d_{(0,0)} f(h,k) = \frac{\partial f}{\partial x}(0,0)\, h + \frac{\partial f}{\partial y}(0,0)\, k = 0$

and $\lim_{\|(h,k)\| \to 0} \frac{f(0,0) - f(h,k) - d_{(0,0)} f(h,k)}{\|(h,k)\|} = 0$

but $\lim_{h \to 0} \frac{f(0,0) - f(h,h) - d_{(0,0)} f(h,h)}{\|(h,h)\|} = \lim_{h \to 0} -\frac{1}{\sqrt{2}} \cdot \frac{h^3}{2h^3} = -\frac{1}{2\sqrt{2}} \neq 0$

Hence f is not differentiable at $(0,0)$

Q16

① f is continuous on $\mathbb{R}^2\setminus\{(0,0)\}$, let's see at $(0,0)$:

$$|f(x,y)| = \left|\frac{xy}{x^2+y^2}\right| \cdot |x^2-y^2| \leq \frac{|x^2-y^2|}{2} \xrightarrow[(x,y)\to(0,0)]{} 0$$

hence $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

Ccl: f is continuous on $\mathbb{R}^2$

② For $(x,y)\neq(0,0)$, $\frac{\partial f}{\partial x}(x,y) = \frac{x^4y-y^5+4x^2y^3}{(x^2+y^2)^2}$

and $\frac{\partial f}{\partial x}(0,0) = 0$

$$\left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)\right| = \left|\frac{x^4y-y^5+4x^2y^3}{(x^2+y^2)^2}\right|$$

$$\leq \frac{|x|^4|y| - |y|^5 + 4|x|^2|y|^3}{(x^2+y^2)^2}$$

I used that
$|x| \leq (x^2+y^2)^{1/2}$
$|y| \leq (x^2+y^2)^{1/2}$

$$\leq 6\frac{(x^2+y^2)^{5/2}}{(x^2+y^2)^2} = 6\sqrt{x^2+y^2} \xrightarrow[(x,y)\to(0,0)]{} 0$$

hence $\frac{\partial f}{\partial x}$ is continuous on $\mathbb{R}^2$

We can prove similarly that $\frac{\partial f}{\partial y}$ is continuous on $\mathbb{R}^2$

③ Hence the partial derivatives of f exist and are continuous on $\mathbb{R}^2$

So f is differentiable on $\mathbb{R}^2$

Q17 ① • $f(x, x^2) = \dfrac{x^4}{2x^4} = \dfrac{1}{2} \xrightarrow[x\to 0]{} \dfrac{1}{2}$

so $\lim_{(x,y)\to(0,0)} f(x,y) \neq 0 = f(0,0)$

and f is not continuous at $(0,0)$

• let $v = (a,b)$ then

$$\partial_v f(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(a,b)) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{f(ta,tb)}{t}$$

$$= \lim_{t\to 0} \frac{t^2a^2tb}{t^5a^4+t^3b^2}$$

$$= \lim_{t\to 0} \frac{a^2b}{t^2a^4+b^2} = \begin{cases} 0 & \text{if } b=0 \\ \frac{a^2}{b} & \text{if } b\neq 0 \end{cases}$$

② • $(e^{-k^2}, 1/k) \xrightarrow[k\to+\infty]{} 0$

but $f(e^{-k^2}, 1/k) = -1 \xrightarrow[k\to+\infty]{} -1 \neq f(0,0)$

hence f is not continuous at $(0,0)$

• let $v = (a,b)$ then

$$\partial_v f(0,0) = \lim_{t\to 0} \frac{f(ta,tb)}{t} = \lim_{t\to 0} \begin{cases} tb^2(\ln|t| - \ln|a|) & \text{if } a\neq 0 \\ 0 & \text{if } a=0 \end{cases} = 0 \text{ in both cases}$$

I used that $\lim_{t\to 0} t\ln|t| = 0$

Q18: ① for $(x,y) \neq (0,0)$ we have:

$$\frac{\partial f}{\partial x}(x,y) = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

which are continuous at $(4,3)$

hence f is differentiable at $(4,3)$ and

$$d_{(4,3)}f(h,k) = \frac{4}{\sqrt{4^2+3^2}}h + \frac{3}{\sqrt{4^2+3^2}}k$$

$$= \frac{4}{5}h + \frac{3}{5}k$$

② $\sqrt{4.05^2+2.93^2} = f\left((4,3)+(0.05,-0.07)\right)$

$$\simeq f(4,3) + d_{(4,3)}f(0.05,-0.07)$$

$$= 5 + \frac{4}{5}0.05 - \frac{3}{5}0.07$$

$$= \frac{25+0.2-0.21}{5}$$

$$= \frac{24.99}{5}$$