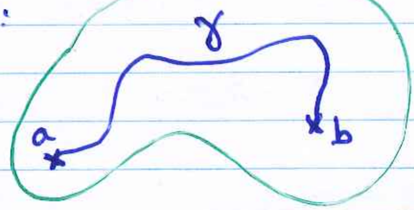
The IVT:

Def. A subset $S \subset \mathbb{R}^m$ is path-connected if $\forall a, b \in S$

$\exists \gamma : [0,1] \to \mathbb{R}^m$ continuous s.t. $\begin{cases} \gamma(0) = a \\ \gamma(1) = b \\ \forall t \in [0,1],\ \gamma(t) \in S \end{cases}$

Ex:



is path-connected

Ex:

is not path-connected

Lemma: the path-connected subsets of $\mathbb{R}$ are the intervals

Δ • An interval is path-connected: let I be an interval and $a, b \in I$ and set $\gamma(t) = (1-t)a + tb$ then γ is continuous and $\begin{cases} \gamma(0) = a \\ \gamma(1) = b \\ \forall t \in [0,1] \\ \gamma(t) \in I \end{cases}$

• A path-connected ~~set~~ subset of $\mathbb{R}$ is an interval: let $S \subset \mathbb{R}$ be path-connected. let $a, c, b \in \mathbb{R}$ s.t. $a < c < b$ and $a, b \in S$.
let γ be a path from a to b.
By the IVT (MAT137), $\exists t_0$ s.t. $\gamma(t_0) = c$ so $c \in S$

$\square$

Theorem: the continuous image of a path-connected set is path-connected
i.e. if $f : \mathbb{R}^m \to \mathbb{R}^k$ is continuous and $S \subset \mathbb{R}^m$ is path-connected then $f(S) \subset \mathbb{R}^k$ is too

Δ let $a, b \in f(S)$. then $a = f(\alpha)$ and $b = f(\beta)$ with $\alpha, \beta \in S$
since S is path-connected, $\exists \tilde{\gamma}$ a path from α to β

then $\gamma = f \circ \tilde{\gamma}$ is a path from a to b

$\square$

Cor (IVT): $f: S \to \mathbb{R}$ continuous and $S \subset \mathbb{R}^m$ path-connected.

if $\exists a, b \in S, \exists t \in \mathbb{R}$ s.t. $f(a) < t < f(b)$

then $\exists c \in S$ s.t. $f(c) = t$

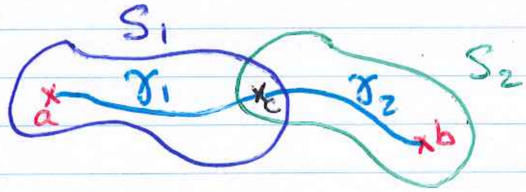
Δ We know that $f(S)$ is path-connected and $\subset \mathbb{R}$ so it is an interval.
Hence $t \in f(S)$ $\square$

Proposition: Let $S_1, S_2 \subset \mathbb{R}^m$ be path-connected

If $S_1 \cap S_2 \neq \phi$ then $S_1 \cup S_2$ is path-connected

↳ ⚠ S_1 S_2 each is path-connected but not their union

Δ



$S = S_1 \cup S_2$

Let $a \in S_1$, $b \in S_2$ we want a path γ on S from a to b
Let $c \in S_1 \cap S_2$
Let $\gamma_1 : [0,1] \to \mathbb{R}^m$ be a path from a to c in S_1
and $\gamma_2 : [0,1] \to \mathbb{R}^m$ ——— c to b in S_2

then $\gamma(t) = \begin{cases} \gamma_1(2t) & \text{if } t \in [0, 1/2] \\ \gamma_2(2t-1) & \text{if } t \in [1/2, 1] \end{cases}$

is a path from a to b in S $\square$

Homework: questions from 1-5 in the online notes