Line integrals

Convention: In this section, by a "curve", I mean a simple regular parametrized C^1 curve, i.e.

$$C = \{\sigma(t) : t \in [a,b]\} \subset \mathbb{R}^m$$

where $\sigma : [a,b] \longrightarrow \mathbb{R}^m$ is C^1 and satisfies
(i) σ is injective on (a,b) "simple points only"
(ii) $\forall t \in (a,b)$, $\sigma'(t) \neq \vec{0}$ "regular"

We say that the curve is closed when $\sigma(a) = \sigma(b)$

Remark: Why these assumptions?

① We don't want C^0 curves to avoid phenomena like space-filling curves

② We don't want differentiable curves but C^1 to ensure that our curves are rectifiable (finite length)

③ We want $\sigma'(t) \neq \vec{0}$ to have regular curves and to ensure that the parametrization captures the geometry of C

$$\{\underbrace{(t,t)}_{= \sigma_1(t)} : t \in [-1,1]\} = \{\underbrace{(t^3,t^3)}_{= \sigma_2(t)} : t \in [-1,1]\}$$

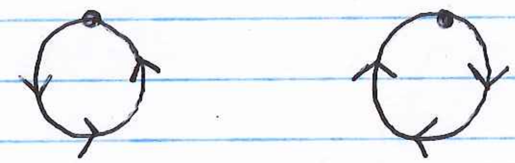
④ We want σ to be injective on (a,b) to have some "correspondance" between the parametrizations and the curves.

eg: $\sigma_1 : [0,1] \longrightarrow \mathbb{R}^2$, $t \mapsto (\cos(2\pi t), \sin(2\pi t))$ and $\sigma_2 : [0,1] \longrightarrow \mathbb{R}^2$, $t \mapsto (\cos(4\pi t), \sin(4\pi t))$ have the same image but not the same length: σ_2 turns twice around O, we don't want that!

Nevertheless, we won't have an exact correspondance:

ⓐ We may have 2 parametrizations with different speeds for the same C

ⓑ We may have 2 parametrizations going in different directions

e.g. 

We say that they have different orientations (there are only two possible orientations)

And this phenomenon will be important for line integrals of vector fields.

⚠ Here I adopted the above conventions from the textbook but notice that usually we define a "curve" in this context as a "parametrization up to diffeomorphisms" and not as the image C.
For this reason we write $\int_\gamma f$ and not $\int_C f$ in this context
This is important in several areas of mathematics (complex analysis, algebraic topology) where we want to distinguish γ_1 and γ_2 from the previous example.
In MAT237 we want our results to only depend on C (up to orientation)
→ ⚠ Be careful if you use other resources/books!

For ②: $\gamma : [-1,1] \longrightarrow \mathbb{R}^2$, $t \mapsto (t, t^2\cos(\pi/t^2))$, $\gamma(0)=(0,0)$ is differentiable, not C^1,

but of infinite length: $\int_{-1}^{1} \|\gamma'(t)\|\, dt = +\infty$

Def: Let $C = \text{Im}\,\sigma = \{\sigma(t) : t \in [a,b]\}$ be a curve (as above) and $f : C \longrightarrow \mathbb{R}$ continuous.

We define $$\int_C f := \int_a^b f(\sigma(t)) \cdot \|\sigma'(t)\|\, dt$$

"the line integral of f along C"

Rem: It doesn't depend on the parametrization σ.
Indeed, let $\sigma_1 : [a,b] \longrightarrow \mathbb{R}^m$ and $\sigma_2 : [c,d] \rightarrow \mathbb{R}^m$ be two parametrizations of C s.t. $\sigma_2 = \sigma_1 \circ \varphi$ where $\varphi : (c,d) \rightarrow (a,b)$ is a C^1-diffeomorphism then

$$\int_c^d f(\sigma_2(t))\, \|\sigma_2'(t)\|\, dt = \int_c^d f(\sigma_1(\varphi(t)))\, \|(\sigma_1 \circ \varphi)'(t)\|\, dt$$

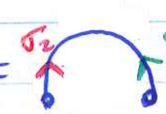
$$= \int_c^d f(\sigma_1(\varphi(t))) \cdot \|\sigma_1'(\varphi(t))\| \cdot |\varphi'(t)|\, dt$$

$$s = \varphi(t) \rightsquigarrow = \int_a^b f(\sigma_1(s)) \cdot \|\sigma_1'(s)\|\, ds$$

⚠ Notice that φ is either increasing ($\varphi'(t) > 0$) or decreasing ($\varphi'(t) < 0$)
It will be useful later!

↳ σ_1, σ_2 have same orientation

or opposite orientation

Ex: $C =$  $\sigma_1(t) = (\cos t, \sin t)$ for $t \in [0, \pi]$
$\sigma_2(t) = (t, \sqrt{1-t^2})$ for $t \in [-1, 1]$

$\sigma_2(t) = \sigma_1(\varphi(t))$ for $\varphi : \begin{array}{ccc} [-1,1] & \longrightarrow & [0,\pi] \\ t & \longmapsto & \arccos(t) \end{array}$

and $\varphi'(t) = -\dfrac{1}{\sqrt{1-t^2}} < 0$

Definition: Let $C = \{\sigma(t) : t \in [a,b]\}$ be a curve as above

We define the arclength of C by

$$\mathcal{L}(C) := \sup_P \left\{ \sum_{j=0}^{m-1} \|\sigma(t_{j+1}) - \sigma(t_j)\| \right\}$$

where P goes through all the partitions

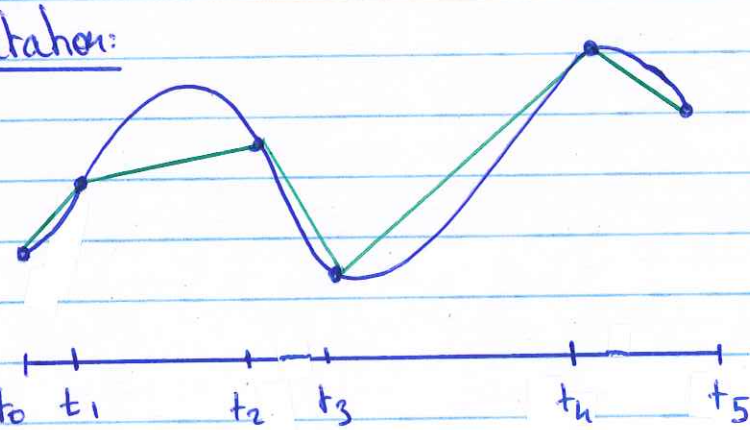
$P = \{a = t_0 < t_1 < \dots < t_m = b\}$ of $[a,b]$

Comment: this supremum is finite (we say that C is rectifiable) since σ is Lipschitz as a C^1-function on $[a,b]$ compact

ie $\exists K \geq 0, \forall t, s \in [a,b], \|\sigma(t) - \sigma(s)\| \leq K|t-s|$

so $\displaystyle\sum_{j=0}^{m-1} \|\sigma(t_{j+1}) - \sigma(t_j)\| \leq \sum_{j=0}^{m-1} K|t_{j+1} - t_j| = K(b-a)$

Geometric interpretation:



The following theorem is very useful since

① it allows to compute $\mathcal{L}(C)$ using integration techniques

② it shows that $\mathcal{L}(C)$ doesn't depend on σ

Theorem: $\boxed{\mathcal{L}(C) = \int_C 1 = \int_a^b \|\sigma'(t)\|\, dt}$

See claim 2 of proof 2 of "A MVT-like inequality"

$\Rightarrow$ Proof : $\boxed{\text{You can skip it}}$

① $\|\sigma(t_{i+1})-\sigma(t_i)\| = \left\| \int_{t_i}^{t_{i+1}} \sigma'(t)\,dt \right\| \leqslant \int_{t_i}^{t_{i+1}} \|\sigma'(t)\|dt$

so $\sum_{i=0}^{m-1} \|\sigma(t_{i+1})-\sigma(t_i)\| \leqslant \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \|\sigma'(t)\|dt = \int_a^b \|\sigma'(t)\|dt$

for any partition, ie $\mathcal{L}(C) \leqslant \int_a^b \|\sigma'(t)\|dt$

② σ $C^1 \Rightarrow \sigma'$ C^0

since $[a,b]$ is compact, σ' is uniformly continuous

ie $\forall \varepsilon>0, \exists \delta>0, \forall t,s \in [a,b], |t-s|<\delta \Rightarrow \|\sigma'(t)-\sigma'(s)\|<\varepsilon$

Let $\varepsilon>0$, take δ as above and P a partition so that $t_{i+1}-t_i < \delta/2$

then $s\in[t_i,t_{i+1}] \Rightarrow \|\sigma'(s)\|-\|\sigma'(t_{i+1})\| \leqslant \|\sigma'(s)-\sigma'(t_{i+1})\| < \varepsilon$

$$\Rightarrow \int_{t_i}^{t_{i+1}} \|\sigma'(s)\|\,ds \leqslant \int_{t_i}^{t_{i+1}} \|\sigma'(t_{i+1})\|+\varepsilon\,ds$$

$b=a+b-a$ $\longrightarrow$ $= \left\| \int_{t_i}^{t_{i+1}} \sigma'(s)+\sigma'(t_{i+1})-\sigma'(s)\,ds \right\| + \varepsilon(t_{i+1}-t_i)$

triangle ineq $\longrightarrow$ $\leqslant \left\| \int_{t_i}^{t_{i+1}} \sigma'(s)\,ds \right\| + \left\| \int_{t_i}^{t_{i+1}} \sigma'(t_{i+1})-\sigma'(s)\,ds \right\| + \varepsilon(t_{i+1}-t_i)$

$\|\sigma'(t_{i+1})-\sigma'(s)\| \leqslant \varepsilon$ $\longrightarrow$ $\leqslant \|\sigma(t_{i+1})-\sigma(t_i)\| + 2\varepsilon(t_{i+1}-t_i)$

$$\Rightarrow \int_a^b \|\sigma'\| = \sum \int_{t_i}^{t_{i+1}} \|\sigma'\| \leqslant \sum \|\sigma(t_{i+1})-\sigma(t_i)\| + 2\varepsilon(b-a)$$
$$\leqslant L(C) + 2\varepsilon(b-a)$$

and then $\varepsilon \longrightarrow 0$

so $\int_a^b \|\sigma'\| \leqslant L(C)$

⚠ In practice $\int \|\sigma'(t)\|\,dt$ may be difficult to compute $\square$

$$\begin{array}{c} \mathbb{R}^{m} \\ \cup \\ F : C \longrightarrow \mathbb{R}^{m} \end{array}$$ same dimension

⚠

Line integral for a vector field

Def: Let $C = \{\sigma(t) : t \in [a,b]\} \subset \mathbb{R}^m$ as above.
Let $\vec{F} : C \longrightarrow \mathbb{R}^m$ be continuous "a vector field"

The line integral of $\vec{F}$ along C is defined by:

$$\underbrace{\int_C \vec{F} \cdot d\vec{x}}_{\text{That's just a notation}} := \int_a^b \underbrace{\vec{F}(\vec{\sigma}(t)) \cdot \vec{\sigma}'(t)}_{\in \mathbb{R}} \, dt$$

$\in \mathbb{R}$ as the dot product of 2 vectors of $\mathbb{R}^m$

Physics notation: $\vec{F} = (F_1, \dots, F_m)$, $\vec{\sigma} = (\sigma_1, \dots, \sigma_m)$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{x} &= \int_a^b \vec{F}(\vec{\sigma}(t)) \cdot \vec{\sigma}'(t) \, dt \\ &= \int_a^b \Big(F_1(\vec{\sigma}(t)) \, \vec{\sigma}_1'(t) + \dots + F_m(\vec{\sigma}(t)) \, \sigma_m'(t) \Big) \, dt \\ &= \sum_{i=1}^{m} \int_a^b \underbrace{F_i(\vec{\sigma}(t))}_{=: F_i} \, \underbrace{\sigma_i'(t) dt}_{=: dx_i} \end{aligned}$$

So a convenient notation / mnemonic device is

$$\int_C \vec{F} \cdot d\vec{x} = \int_C F_1 \, dx_1 + \dots + F_m \, dx_m$$

where $\displaystyle\int_C F_i \, dx_i := \int_a^b F_i(\vec{\sigma}(t)) \, \sigma_i'(t) \, dt$

⚠ That's just a notation, for instance

$$\int_C -y \, dx + x \, dy = \int_C \vec{F} \cdot d\vec{x} = \int_a^b \vec{F}(\vec{\sigma}(t)) \cdot \vec{\sigma}'(t) \, dt$$

where $\vec{F}(x,y) = (-y, x)$

Do NOT try to compute $\int -y \, dx$ directly, it's a notation!!!
! that's all !

⚠ Orientation dependence of $\int_C \vec{F} \cdot d\vec{x}$

Let $\sigma_1 : [a,b] \longrightarrow \mathbb{R}^n$, $\sigma_2 : [c,d] \longrightarrow \mathbb{R}^n$ as above such that
$C = \operatorname{Im} \sigma_1 = \operatorname{Im} \sigma_2$

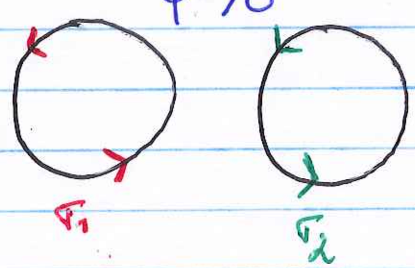
then $\sigma_2 = \sigma_1 \circ \varphi$ where $\varphi : (c,d) \longrightarrow (a,b)$ is a C^1-diffeomorphism

Notice that φ is either always increasing $(\varphi' > 0)$
or always decreasing $(\varphi' < 0)$ ⚠

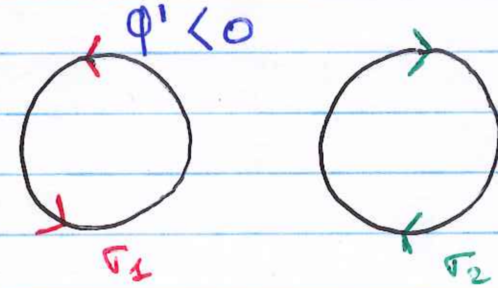
If $\varphi' > 0$ we say that σ_1 and σ_2 have <u>same</u> orientation

If $\varphi' < 0$ we say that σ_1 and σ_2 have <u>opposite</u> orientation

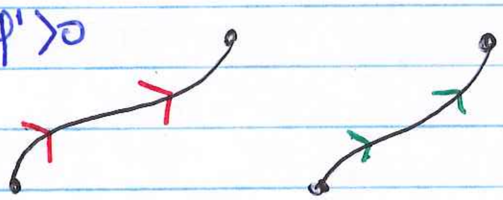
(1) $\varphi' > 0$



(2) $\varphi' < 0$



(3) $\varphi' > 0$



(4) $\varphi' < 0$

Let's see what happen to $\int_C \vec{F} \cdot d\vec{x}$:

$$\int_c^d \vec{F}(\vec{\sigma}_2(t)) \cdot \vec{\sigma}_2'(t)\, dt = \int_c^d \vec{F}(\vec{\sigma}_1(\varphi(t))) \cdot \left(\varphi'(t)\, \vec{\sigma}_1'(\varphi(t))\right) dt$$

since the dot products is linear and $\varphi' \in \mathbb{R}$
$$= \int_c^d \left(\vec{F}(\vec{\sigma}_1(\varphi(t))) \cdot \vec{\sigma}_1'(\varphi(t))\right) \underbrace{\varphi'(t)}\, dt$$

there is absolute value for the CoV

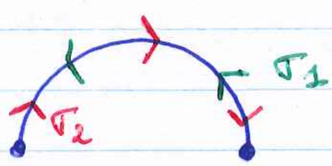
$$= \begin{cases} + \int_a^b \vec{F}(\vec{\sigma}_1(s)) \cdot \vec{\sigma}_1'(s)\, ds & \text{if } \varphi' > 0 \\ \text{or} \\ - \int_a^b \vec{F}(\vec{\sigma}_1(s)) \cdot \vec{\sigma}_1'(s)\, ds & \text{if } \varphi' < 0 \end{cases}$$

Conclusion :

$$\int_C^d \vec{F}(\vec{\gamma_2}(t)) \cdot \vec{\gamma_2}'(t)\, dt = \begin{cases} \int_a^b \vec{F}(\vec{\gamma_1}(t)) \cdot \vec{\gamma_1}'(t)\, dt & \text{if } \gamma_1 \text{ and } \gamma_2 \text{ have same orientation} \\ -\int_a^b \vec{F}(\vec{\gamma_1}(t)) \cdot \vec{\gamma_1}'(t)\, dt & \text{if } \gamma_1 \text{ and } \gamma_2 \text{ have opposite orientation} \end{cases}$$

⚠ $\int_C \vec{F} \cdot d\vec{x}$ depends on the orientation of γ (and not only on its image $C = \mathrm{Im}\,\gamma$)

Ex : $\gamma_1 : \begin{array}{l}[0,\pi] \longrightarrow \mathbb{R}^2 \\ t \longmapsto (\cos t, \sin(t))\end{array}$ $\quad \gamma_2 : \begin{array}{l}[-1,1] \longrightarrow \mathbb{R}^2 \\ t \longmapsto (t, \sqrt{1-t^2})\end{array}$

$C =$ 

$\gamma_2 = \gamma_1 \circ \varphi$

$\varphi(t) = \arccos t$, $\varphi'(t) = -\frac{1}{\sqrt{1-t^2}} < 0$

So γ_1, γ_2 have opposite orientation

① $\int_C -y\,dx + x\,dy$ with γ_1 :

$$\int_C -y\,dx + x\,dy = \int_0^\pi \underbrace{(-\sin t)}_{-y}\underbrace{(-\sin t)}_{dx} + \underbrace{(\cos t)}_{x}\underbrace{(\cos t)}_{dy}\, dt$$

$$= \int_0^\pi \sin^2 t + \cos^2 t\, dt = \int_0^\pi dt = \pi$$

② $\int_C -y\,dx + x\,dy$ with γ_2 :

$$\int_C -y\,dx + x\,dy = \int_{-1}^1 -\sqrt{1-t^2} - t\,\frac{t}{\sqrt{1-t^2}}\, dt = -\int_{-1}^1 \frac{1-t^2+t^2}{\sqrt{1-t^2}}\, dt$$

$$= -\int_{-1}^1 \frac{1}{\sqrt{1-t^2}}\, dt = -\left[\arcsin(t)\right]_{-1}^1 = -\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$= -\pi$$

When $\vec{F}$ is a gradient field, ie $\vec{F} = \nabla f$ for some $f : \mathbb{R}^n \to \mathbb{R}$ C^1 then $\int_C \vec{F} \cdot d\vec{x}$ is easy to compute as showed in the next theorem

In physics we say that the vector field $\vec{F}$ is conservative when $\vec{F} = \nabla f$ for some f C^1

Theorem: (Gradient theorem / FTC for line integrals)

Let $C = \{\sigma(t) : t \in [a,b]\} \subset \mathbb{R}^m$ be a curve as above

Let $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ be C^1, set $F : \mathbb{R}^m \longrightarrow \mathbb{R}^m$, $x \longmapsto \nabla f(x)$

then
$$\int_C \vec{F} \cdot d\vec{x} = f(\sigma(b)) - f(\sigma(a))$$

or in a more concise way: $\int_C \vec{\nabla f} \cdot d\vec{x} = f(\sigma(b)) - f(\sigma(a))$

⚠ If $\vec{F} = \nabla f$ then $\int_C \vec{F} \cdot d\vec{x}$ only depends on the values of f at the endpoints

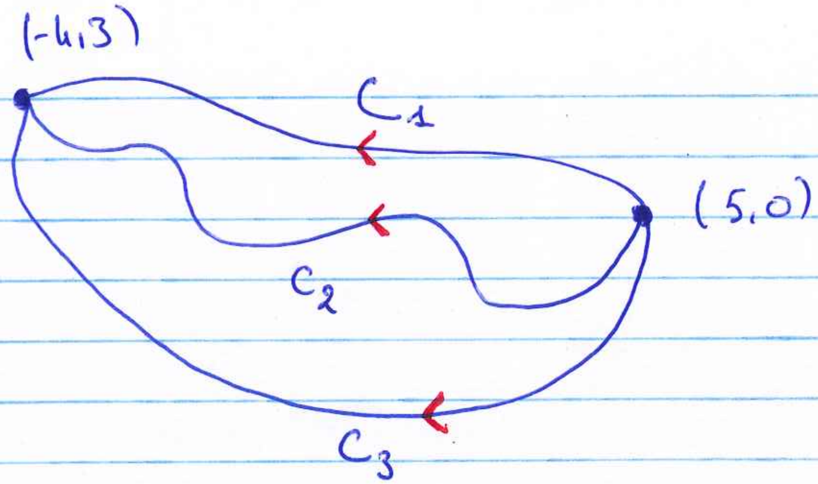
$$\Delta \quad \int_C \vec{F} \cdot d\vec{x} = \int_a^b \nabla f(\sigma(t)) \cdot \sigma'(t)\, dt = \int_a^b (f \circ \sigma)'(t)\, dt$$
$$= [f \circ \sigma(t)]_a^b \overset{\text{FTC}}{=} f(\sigma(b)) - f(\sigma(a)) \qquad \square$$

Corollary: If $C = \{\sigma(t) : t \in [a,b]\} \subset \mathbb{R}^m$ is closed (ie $\sigma(a) = \sigma(b)$) and $\vec{F} = \nabla f$ for $f : \mathbb{R}^m \to \mathbb{R}$ C^1

then
$$\int_C \vec{F} \cdot d\vec{x} = 0$$

$$\Delta \quad \int_C \vec{F} \cdot d\vec{x} = f(\sigma(b)) - f(\sigma(a)) = f(p) - f(p) = 0 \quad \text{where } p = \sigma(a) = \sigma(b) \qquad \square$$

Ex :



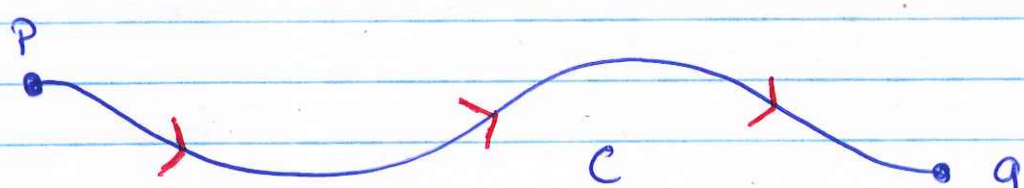
Compute $\int_{C_i} y\,dx + x\,dy = \int_{C_i} \vec{F} \cdot d\vec{x}$

where $\vec{F}(x,y) = (y, x)$

Notice that $\vec{F} = \nabla f$ where $f(x,y) = xy$

So $\int_{C_i} y\,dx + x\,dy = f(-4,3) - f(5,0) = -4 \cdot 3 - 5 \cdot 0 = -12$

Ex :



$\alpha \geq 1$

Compute $\int_C \|\vec{x}\|^{\alpha-1} \vec{x} \cdot d\vec{x}$ ie $\vec{F}(\vec{x}) = \|\vec{x}\|^{\alpha-1} \vec{x}$

Notice that $\vec{F} = \nabla f$ where $f(\vec{x}) = \dfrac{\|\vec{x}\|^{\alpha+1}}{\alpha+1}$

So $\int_C \|\vec{x}\|^{\alpha-1} \vec{x} \cdot d\vec{x} = f(q) - f(p) = \dfrac{\|q\|^{\alpha+1} - \|p\|^{\alpha+1}}{\alpha+1}$

Ex: $\int_C \|\vec{x}\|^{-2} \vec{x} \cdot d\vec{x} = \log\|q\| - \log\|p\|$

Since $\|\vec{x}\|^{-2} \vec{x} = \nabla(\log\|\vec{x}\|)$