Improper integrals

Comment: I strongly suggest you to review the material from MAT137 in the one-variable case
I posted some notes online

Goal: What can we do if a function or its domain is not bounded?

Disclaimer: I use a slightly different approach from the textbook
It is equivalent and it doesn't change anything in practice.

Definition: $\mathcal{U} \subset \mathbb{R}^n$ open, $f : \mathcal{U} \to \mathbb{R}$ non-negative

We say that $\int_{\mathcal{U}} f$ is convergent (or that f is improperly integrable on $\mathcal{U}$)

when ① $\forall S \subset \mathcal{U}$ compact and Jordan measurable (∂S has ZC)

$$\int_S f \text{ exists}$$

② $\sup_S \int_S f < +\infty$ where S are like above

And then we set $\int_{\mathcal{U}} f := \sup_S \int_S f$

Fact: For $\mathcal{U} \subset \mathbb{R}^n$ open, there exists a sequence $C_1, \ldots, C_i, \ldots$ of compact Jordan measurable subsets of $\mathcal{U}$ s.t.

① $\mathcal{U} = \bigcup C_i$

② $C_i \subset \mathring{C}_{i+1}$

We say that $(C_i)_i$ is an exhaustion of $\mathcal{U}$

Ex: • $\mathcal{U} = \mathbb{R}^n$, $C_i = B(\vec{0}, i)$

• $\mathcal{U} = \mathbb{R}^n \setminus \{0\}$, $C_i = \left\{ x \in \mathbb{R}^n : \frac{1}{i} \leq \|x\| \leq i \right\}$

Theorem: $\mathcal{U} \subset \mathbb{R}^n$ open, $f : \mathcal{U} \to \mathbb{R}$ non-negative, $\int_S f$ exists for S compact Jordan measurable
$(C_i)_{i \in \mathbb{N}}$ as above

$$\int_{\mathcal{U}} f \text{ is convergent } \Leftrightarrow \exists M \geq 0, \forall i, \int_{C_i} f < M$$

and then $\int_{\mathcal{U}} f = \lim_{i \to +\infty} \int_{C_i} f$

Δ $u_i = \int_{C_i} f$ is non-decreasing so it converges iff $\left\{ \int_{C_i} f \right\}$ is bounded

$\Rightarrow$: $\int_{C_i} f \leq \int_{\mathcal{U}} f = \sup_S \int_S f$

$\Leftarrow$: Let $S \subset \mathcal{U}$ compact Jordan measurable.
by compactness $S \subset \bigcup_{i=1}^{K} \mathring{C_i} \subset \bigcup_{i=1}^{K} C_i$ (finitely many)

so $\int_S f \leq \int_{C_K} f \leq \lim_{i \to +\infty} \int_{C_i} f$

$\square$

The above theorem is the reason why I first restricted to the non-negative case:
if (C_i) is an exhausting approximation of $\mathcal{U}$ then the value of $\lim_{i \to +\infty} \int_{C_i} f$ doesn't depend on the choice of C_i

That's false for f not non-negative.

Cl: if $f \geq 0$ then you will always get the same value for $\int_{\mathcal{U}} f$, no matter how you try to compute it

Ex : in the one variable case, we define $\int_a^{+\infty} f$ by $\lim_{c \to +\infty} \int_a^c f$

so that $\displaystyle\int_0^{+\infty} \frac{\sin(x)}{x}\,dx = \lim_{m \to +\infty} \int_0^m \frac{\sin(x)}{x}\,dx = \lim_{m \to +\infty} \int_{[0,m]} \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$

where we used $C_m = [0,m]$ (that's the convention in the 1-var case)

But for another choice of C_m, we could have obtained something else

$$D_m = [0, 2m\pi - \pi] \cup \bigcup_{k=m}^{2m} [2k\pi, 2k\pi + \pi]$$

then $$\int_{D_m} \frac{\sin(x)}{x}\,dx = \int_0^{2m\pi - \pi} \frac{\sin(x)}{x}\,dx + \sum_k \underbrace{\int_{2k\pi}^{2k\pi+\pi} \frac{\sin(x)}{x}\,dx}_{\geqslant \frac{1}{2k\pi+\pi} \int \sin(x)\,dx}$$

so $$\lim_{m \to +\infty} \int_{D_m} \frac{\sin(x)}{x}\,dx > \frac{\pi}{2} + \lim_{m\infty} \sum_{k=m}^{2m} \frac{1}{k+1} = \frac{\pi}{2} + \frac{\ln 2}{\pi} > \frac{\pi}{2}$$

Ex : $$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\,dy\,dx = \int_0^1 \left[\frac{y}{x^2+y^2}\right]_0^1 dx = \int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$$

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\,dx\,dy = \int_0^1 \left[-\frac{x}{x^2+y^2}\right]_0^1 dy = \int_0^1 -\frac{1}{1+y^2}\,dy = -\frac{\pi}{4}$$

ie with $C_m = [1/m, 1] \times [0,1]$ we get $\pi/4$

with $C_m = [0,1] \times [1/m, 1]$ we get $-\pi/4$

and we can prove that with $[1/m, 1] \times [1/m, 1]$ we get 0. (Do it)

What should we do when f is not non-negative?

We will see that if $\int_\Omega |f|$ is CV then $\int_\Omega f$ is CV and doesn't depend on choices. So we will only work with absolute CV!!!

Notice that you already met this phenomenon for series in MAT137

① If $\sum a_n$ is abs cv (ie $\sum |a_n| < +\infty$) then you can compute $\sum a_n$ by grouping terms or permuting terms and you will always get the same value

② If $\sum a_n$ is cv but not ACV, by Riemann rearrangement theorem, you can get any value (even $+\infty$ or $-\infty$) by permuting terms

Let's go back to integrals.

__Notation__: for $f: U \to \mathbb{R}$, $U \subset \mathbb{R}^n$, we define $f_+, f_-: U \to \mathbb{R}$ by $f_+(x) = \max(f(x), 0)$ and $f_-(x) = \max(-f(x), 0)$

so that $f_+, f_- \geq 0$ and $f = f_+ - f_-$ and $|f| = f_+ + f_-$

__Definition__: $U \subset \mathbb{R}^n$ open, $f: U \to \mathbb{R}$
We say that $\int_U f$ is (absolutely) convergent or that f is improperly integrable on U if f_+ and f_- are integrable on U and then we set

$$\int_U f := \int_U f_+ - \int_U f_-$$

__Remark__: this definition is equivalent to the one in the textbook! The absolute CV aspect is hidden in the fact that S can be any Jordan measurable set and not only a ball!

Theorem: $f: \mathcal{U} \to \mathbb{R}$, $\mathcal{U} \subset \mathbb{R}^n$ open, $\int_S f$ exists for any $S \subset \mathcal{U}$ Jordan measurable

$\int_{\mathcal{U}} f$ is improperly integrable if and only if $\int_{\mathcal{U}} |f|$ is integrable

⚠ that's different from the one-variable case where $\int_0^{+\infty} \frac{\sin x}{x}$ exists!

Δ $\Rightarrow$ if f is improperly integrable then f_+ and f_- are too by definition but then $|f| = f_+ + f_-$

If S is compact Jordan measurable:

$$\int_S |f| = \int_S f_+ + \int_S f_- \leqslant \int_{\mathcal{U}} f_+ + \int_{\mathcal{U}} f_-$$

so $\sup \int_S |f| < +\infty$

this assumption is important!

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{otherwise} \end{cases}$$

⚠ $\int_S \frac{|f| + f}{2}$ exists since $\int_S f$ exists

$\Leftarrow$: $0 \leqslant f_+ \leqslant f_+ + f_- = |f|$

so if S is Jordan measurable compact $\int_S f_+ \leqslant \int_S |f| \leqslant \int_{\mathcal{U}} |f|$

and similarly for f_- $\square$

The following theorem ensures that if f is integrable then we can compute $\int_{\mathcal{U}} f$ without being too careful since we force $\int f$ to be ACV in our definition!

Theorem: Assume that f is integrable on $\mathcal{U}$ and let C_i be a sequence of compact Jordan measurable sets s.t.

① $\mathcal{U} = \bigcup_i C_i$ ② $C_i \subset \mathring{C}_{i+1}$

then $$\int_{\mathcal{U}} f = \lim_{i \to +\infty} \int_{C_i} f$$

$$\Delta \int_{\mathcal{U}} f = \int_{\mathcal{U}} f_+ - \int_{\mathcal{U}} f_- = \lim_i \int_{C_i} f_+ - \lim_i \int_{C_i} f_-$$

$$= \lim_i \int_{C_i} f_+ - f_- = \lim_i \int_{C_i} f \quad \square$$

(Connection with the definition from the textbook)

Theorem: $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ s.t. $\forall S \subset \mathbb{R}^n$ Jordan measurable $\int_S$ exists

$\int f$ is improperly convergent

$\iff \exists L > 0, \forall \varepsilon > 0, \exists r > 0, \forall S \subset \mathbb{R}^n$ Jordan measurable

$\bar{B}(0,r) \subset S \Rightarrow \left| \int_S f - L \right| < \varepsilon$

$\Delta \Rightarrow$ you can work separately with f_+, f_- which are non-negative and then $\left| \int_S f - \int_{\mathbb{R}^n} f_+ - \int_{\mathbb{R}} f_- \right| \leq \left| \int_S f_+ - \int_{\mathbb{R}^n} f_+ \right| + \left| \int_S f_- - \int_{\mathbb{R}^n} f_- \right|$

$\Leftarrow$ Claim 1: if (C_i) is an exhaustion then
$\lim_{i \to +\infty} \int_{C_i} f = L$

Δ By Heine-Borel, $\bar{B}(0,r) \subset \bigcup_{i=1}^{K} C_i = C_K$

so $\forall i > K$, $\left| \int_{C_i} f - L \right| < \varepsilon$ $\square$

Claim 2: if $\lim \int_{C_i} f$ doesn't depend on the exhaustion then $\int |f|$ is integrable

Δ Idea: by contradiction either $\int_{\mathbb{R}^n} f_+ = +\infty$ or $\int_{\mathbb{R}^n} f_- = +\infty$

since $f_+ f_- = 0$ we may find an exhaustion that makes $\int f_+$ or $\int f_-$ fast enough so that the other can't compensate.. see the example for $\int_{D_n} \frac{\sin(x)}{x}$ $\square$

the actual proof of Claim 2 relies on the fact that if $\int_S f$ exists for S Jordan measurable, $\exists T \subset S$ Jordan measurable s.t. $\int_S |f| \leq 3 \left| \int_T f \right|$ and we can prove that the latter is bounded $\square$

Formal proof of Claim 2: (You can safely skip it)

Lemma 1: Claim 1 $\Rightarrow \exists M \geq 0, \forall S$ Jordan measurable, $|\int_S f| \leq M$

Δ Assume $\{\int_S f\}$ is not bounded and let (C_n) an exhaustion

for $n \in \mathbb{N}$, $\exists T_n$ s.t. $|\int_{T_n} f| \geq n + \int_{U_n} |f|$, $U_n = C_n \cup T_1 \cup \dots \cup T_{n-1}$

Define $S_n = T_n \cup U_n$ so that (S_n) is an exhaustion

then $|\int_{S_n} f| = |\int_{T_n} f + \int_{S_n \setminus T_n} f| \geq |\int_{T_n} f| - \int_{U_n} |f| \geq n \xrightarrow[n \to +\infty]{} +\infty$

$|L| = +\infty$ contradiction

$\square$

Lemma 2: $S \subset \mathbb{R}^n$, Jordan measurable, $\int_S f$ integrable
then $\exists T \subset S$ Jordan measurable s.t. $\int_S |f| \leq 3 |\int_T f|$

Δ $\delta := \int_S |f| = \int_S f_+ + \int_S f_-$ exists by assumption, so either $\int_S f_+$ or $\int_S f_- \geq \frac{\delta}{2}$

Let say $\int_S f_+ \geq \delta/2 > \delta/3$ (I assume $\delta > 0$ otherwise the Lemma is trivial)

So $\exists R \supset S$ a rectangle with a partition so that $L_P(f_+) > \delta/3$
Define T as the union of the subrectangles where $\inf(f_+) > 0$
then $\int_T f = \int_T f_+ > 0$ $\qquad$ is extended by 0

Here I assume f is C^0 so if $\inf f_+ > 0$ then $f > 0$ on the rectangle
that's not a big assumption : so this proof works if the discontinuity set has $Z \subset$

then $|\int_T f| = \int_T f = \int_T f_+ = \int_R f_+ > \delta/3 = \frac{1}{3} \int_S |f|$

$\square$

Proof of claim 2: $\exists M$ s.t. $\forall T$, $|\int_T f| \leq M$ by Lemma 1
so $\forall S$, $\int_S |f| \leq 3M$ and $\{\int_S |f|\}$ is bounded

$\square$

What do we do in practice?

Step 0: check that $\int_S f$ exists for S Jordan measurable (ex: f C^0)

Step 1: compute $\int_U |f|$

since $|f| \geq 0$, you don't have to be careful, the result won't depend on how you compute it

For instance you can compute $\lim_{r \to +\infty} \int_{B(\vec{0},r)} |f|$ if $U = \mathbb{R}^n$

Case 1: $\int_U |f| = +\infty$: you stop... f is not integrable on U

Case 2: $\int_U |f| < +\infty$: f is integrable, go to step 2

Step 2: we know that $\int_U f$ is (absolutely) integrable

then we can compute it (the result won't depend on how you compute it: every operation that seems legit is legit. for instance you can compute $\lim \int_{B(0,r)} f$ is $U = \mathbb{R}^n$ or compute the $\int dx \int dy$ separately...)

Basic properties: $U \subset \mathbb{R}^n$ open, $f, g : U \to \mathbb{R}$ improperly integrable then:

① $af + bg$ is improperly integrable and $\int af + bg = a\int f + b\int g$

② $\forall x \in U$, $f(x) \leq g(x) \Rightarrow \int_U f \leq \int_U g$

③ $|f|$ is integrable on U and $\left|\int_U f\right| \leq \int_U |f|$

Δ ① $|af+bg| \leqslant |a|\,|f| + |b|\,|g|$

and if S is compact Jordan measurable

$$\int_S |af+bg| \leqslant |a| \int_S |f| + |b| \int_S |g| \leqslant |a| \int_U |f| + |b| \int_U |g|$$

then we obtain the equality by computing $\lim_i \int_{C_i} af+bg$

② For C_i compact and Jordan measurable

$$\int_{C_i} f \leqslant \int_{C_i} g$$ and then take the limit

③ $$\left.\begin{array}{l} f \leqslant |f| \Rightarrow \int f \leqslant \int |f| \\ -f \leqslant |f| \Rightarrow -\int f \leqslant \int |f| \end{array}\right\} \Rightarrow \left|\int f\right| \leqslant \int |f|$$

$\square$

Improper integrals : examples

① A first example with unbounded domain

We want to compute $\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dxdy$

Since $f(x,y) = e^{-(x^2+y^2)}$ is continuous and non-negative

$\lim\limits_{n\to+\infty} \int_{C_n} e^{-(x^2+y^2)}$ doesn't depend on the choice of the exhaustion

(but it could be $+\infty$)

$$\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy = \lim_{n\to+\infty} \int_{\bar{B}(0,n)} e^{-(x^2+y^2)}\,dxdy$$

$$= \lim_{n\to+\infty} \int_{[0,n]\times[-\pi,\pi]} r e^{-r^2}\,dr\,d\theta$$

$$= \lim_{n\to+\infty} 2\pi \left[-\frac{e^{-r^2}}{2}\right]_0^n$$

$$= \pi$$

$$\boxed{\int_{\mathbb{R}^2} e^{-(x^2+y^2)} = \pi}$$

Now let's take another exhaustion:

$$\pi = \int_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dxdy = \lim_{n\to+\infty} \int_{[-n,n]\times[-n,n]} e^{-(x^2+y^2)}\,dxdy = \lim_{n\to+\infty} \int_{-n}^{n} e^{-x^2}\,dx \int_{-n}^{n} e^{-y^2}$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2$$

So : $\boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}}$ (Gaussian/Euler-Poisson integral)

Exercise : $\int_{\mathbb{R}^n} e^{-\alpha\|x\|^2} = \left(\frac{\pi}{\alpha}\right)^{n/2}$

$\alpha > 0$

(2) Define $f : \mathbb{R}^m \setminus \{0\} \longrightarrow \mathbb{R}$ by $f(x) = \dfrac{1}{\|x\|^p}$

(i) $\displaystyle\int_{\{\|x\| < a\}} \|x\|^{-p}$ is convergent $\Leftrightarrow p < m$ (Unbounded function)

(ii) $\displaystyle\int_{\{\|x\| > a\}} \|x\|^{-p}$ is convergent $\Leftrightarrow p > m$ (Unbounded domain)

Δ I prove it for $m = 2$ here (see next page for the general case)

Since f is C^0 and non-negative:

(i) $$\int_{\{\|x\| < a\}} \|x\|^{-p} = \lim_{k \to +\infty} \int_{\{\frac{1}{k} \leq \|x\| \leq a\}} \|x\|^{-p} = \lim_{k \to +\infty} \int_{[1/k, a] \times [-\pi, \pi]} r^{1-p} \, dr \, d\theta$$

$$= \lim_{k \to +\infty} 2\pi \int_{1/k}^{a} \frac{1}{r^{p-1}} dr$$

$$= \begin{cases} +\infty & \text{if } p-1 \geq 1 \\ < +\infty & \text{if } p-1 < 1 \end{cases}$$

(ii) $$\int_{\{\|x\| > a\}} \|x\|^{-p} = \lim_{k \to +\infty} \int_{\{\|x\| \leq k\}} \|x\|^{-p} = \lim_{k \to +\infty} \int_{[0, k] \times [-\pi, \pi]} r^{1-p} \, dr \, d\theta$$

$$= \lim_{k \to +\infty} 2\pi \int_0^k \frac{1}{r^{p-1}} dr = \begin{cases} +\infty & \text{if } p-1 \leq 1 \\ < +\infty & \text{if } p-1 > 1 \end{cases}$$

$\square$

A useful corollary:

<u>Theorem</u>: $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ C^0

if $\exists C > 0, p > m$ s.t. $\|x\|^p |f(x)| \leq C$ then $\displaystyle\int_{\mathbb{R}^m} f$ exists

General case: $m \geqslant 2$

Spherical coordinates in $\mathbb{R}^m$: $(r, \theta, \varphi_1, \dots, \varphi_{m-2})$, $r > 0$, $\theta \in (0, 2\pi)$, $\varphi_i \in (0, \pi)$

$$x_1 = r \cos \varphi_1$$
$$x_2 = r \sin\varphi_1 \cos\varphi_2$$
$$x_3 = r \sin\varphi_1 \sin\varphi_2 \cos\varphi_3$$
$$\dots$$
$$x_{m-1} = r \sin\varphi_1 \cdots \sin\varphi_{m-2} \cos\theta$$
$$x_m = r \sin\varphi_1 \cdots \sin\varphi_{m-2} \sin\theta$$

Notice that $r = \|x\| = \sqrt{\sum x_i^2}$ and that $\left|\det D\Phi\right| = r^{m-1} \prod_{i=1}^{m-2} \sin^{m-i-1}(\varphi_k)$

$$= r^{m-1} \sin^{m-2}(\varphi_1) \sin^{m-3}(\varphi_2) \cdots \sin(\varphi_{m-2})$$

So that: $$\int_{\{\|x\| > a\} \subset \mathbb{R}^m} \|x\|^{-p} = \lim_{k \to +\infty} \int_0^{\pi} \cdots \int_0^{\pi} \int_0^{2\pi} \int_a^k r^{m-p-1} \prod \sin^{m-i-1}(\varphi_k)\, dr\, d\theta\, d\varphi_1 \dots d\varphi_{m-2}$$

$$= \lim_{k \to +\infty} C \int_a^k r^{m-p-1}\, dr = \begin{cases} +\infty & \text{if } p+1-m \leqslant 1 \\ < +\infty & \text{if } p+1-m > 1 \end{cases}$$

(3) A first non-improperly-integrable function (Unbounded domain)

We want to know if $\displaystyle\int_{\{x>0,y>0\}} \sin(x^2+y^2)\,dxdy$ is convergent

$$\lim_{k\to+\infty} \int_{\{x^2+y^2\leq k^2,\, x\geq 0,\, y\geq 0\}} \sin(x^2+y^2)\,dxdy = \lim_{k\to+\infty} \int_0^k \int_0^{\pi/2} r\sin(r^2)\,dr\,d\theta$$

$$= \frac{\pi}{4} \lim_{k\to+\infty} \int_0^{k^2} \sin(u)\,du$$

$$= \frac{\pi}{4} \lim_{k\to+\infty} \left(1-\cos(k^2)\right) \text{ DNE}$$

So $\displaystyle\int_{\{x>0,y>0\}} \sin(x^2+y^2)$ is not improperly convergent.

Here we were able to conclude because we found an exhaustion so that the limit DNE
If we had found a well-defined limit that wouldn't have been enough to conclude: a function whose sign changes which is not improperly CV can be cv for some exhaustion.
See the following example

$$\lim_{k\to+\infty} \int_{[0,k]\times[0,k]} \sin(x^2+y^2)\,dxdy = \lim_{k\to+\infty} \int_{[0,k]\times[0,k]} \sin(x^2)\cos(y^2)+\cos(x^2)\sin(y^2)\,dxdy$$

Fresnel integrals, we admit it

$$= \lim_{k\to+\infty} 2\int_0^k \sin(x^2)\,dx \int_0^k \cos(y^2)\,dy = 2\sqrt{\frac{\pi}{8}}\sqrt{\frac{\pi}{8}} = \frac{\pi}{4}$$

But we have already seen that it is not CV
(so the value depends on the choice)

④ An example where the sign changes (unbounded function)

We want to know whether $\int_{\{x^2+y^2<4\}} \log\left((x^2+y^2)^{1/2}\right) dx dy$ converges

⚠ Here the sign changes, so it is $\boxed{\text{not}}$ enough to check the limit on an exhaustion to prove the improper convergence

$$\lim_{k\to+\infty} \int_{\{1/k\leq x^2+y^2\leq 4\}} \left|\log\sqrt{x^2+y^2}\right| dx dy$$

$$= \lim_{k\to+\infty} \int_0^{2\pi}\int_{1/k}^{2} r|\log r|\, dr d\theta$$

$$= \lim_{k\to+\infty} 2\pi \int_{1/k}^{2} r|\log r|\, dr$$

$$= \lim_{k\to+\infty} 2\pi\left(\int_{1/k}^{1} r(-\log r)dr + \int_1^2 r\log r\, dr\right)$$

$$= \lim_{k\to+\infty} -2\pi\left(\left[\frac{1}{2}r^2\log r - \frac{1}{4}r^2\right]_{1/k}^{1} - \left[1/2\, r^2\log r - \frac{1}{4}r^2\right]_1^2\right)$$

$$= \pi(\log 16 - 1) < +\infty$$

So $\int_{\{x^2+y^2<4\}} \log\sqrt{x^2+y^2}\, dx dy$ is improperly convergent

⑤ Study $\displaystyle\int_{(0,1)\times(0,1)} \frac{1}{x^2y^2}\,dxdy$ (Unbounded function)

Since $f(x,y) = \dfrac{1}{x^2y^2}$ is non-negative and C^0

$$\int_{(0,1)\times(0,1)} \frac{1}{x^2y^2}\,dxdy = \lim_{k\to+\infty} \int_{[1/k,1]\times[1/k,1]} \frac{1}{x^2y^2}\,dxdy$$

$$= \lim_{k\to+\infty} \int_{1/k}^{1} \frac{1}{x^2}\,dx \int_{1/k}^{1} \frac{1}{y^2}\,dy$$

$$= \lim_{k\to+\infty} (1-k)^2 = +\infty$$

So $\displaystyle\int_{(0,1)\times(0,1)} \frac{1}{x^2y^2}\,dxdy$ is not improperly convergent

⑥ Study $\displaystyle\int_{(1,\infty)\times(1,\infty)} \frac{1}{x^2y^2}\,dxdy$ (Unbounded domain)

Since f is continuous and non-negative

$$\int_{(1,\infty)\times(1,\infty)} \frac{1}{x^2y^2}\,dxdy = \lim_{k\to+\infty} \int_{[1,k]\times[1,k]} \frac{1}{x^2y^2}\,dxdy$$

$$= \lim_{k\to+\infty} \left(1-\frac{1}{k}\right)^2 = 1$$

So $\displaystyle\int_{(1,\infty)\times(1,\infty)} \frac{1}{x^2y^2}\,dxdy$ is improperly CV

(7) Study $\int_{\mathbb{R}^2} \frac{1}{1+x^2+y^2}\,dxdy$ (Unbounded domain)

Since $f(x,y) = \frac{1}{1+x^2+y^2}$ is C^0 and non-negative

$$\int_{\mathbb{R}^2} \frac{1}{1+x^2+y^2}\,dxdy = \lim_{k\to+\infty} \int_{\overline{B}(0,k)} \frac{1}{1+x^2+y^2}\,dxdy$$

$$= \lim_{k\to+\infty} \int_{[0,k]\times[-\pi,\pi]} \frac{1}{1+r^2}\,r\,dr\,d\theta$$

$$= \lim_{k\to+\infty} 2\pi \int_0^k \frac{r}{1+r^2}\,dr$$

$$= \lim_{k\to+\infty} \pi \ln(1+k^2) = +\infty$$

So $\int_{\mathbb{R}^2} \frac{1}{1+x^2+y^2}\,dxdy$ is not improperly CV

(I like this one because it looks like $\int_{\mathbb{R}} \frac{1}{1+x^2}\,dx$ which is improperly CV)