University of Toronto – MAT137Y1 – LEC0501

Calculus!

Solutions for slide 6

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Disclaimer: those are *quick-and-dirty* notes written just after the class, so it is very likely that they contain some mistakes/typos...

5. $\displaystyle\sum_{n=1}^{+\infty} \frac{n}{3^n}$

   Notice that $\displaystyle\sum_{n=1}^{+\infty} \frac{n}{3^n} = \sum_{n=1}^{+\infty} n\left(\frac{1}{3}\right)^n = S_5\left(\frac{1}{3}\right)$ where $S_5(x) = \displaystyle\sum_{n=1}^{+\infty} nx^n$ has a radius of convergence equal to 1 (check it).
   Hence for $x \in (-1, 1)$ we have:

$$S_5(x) = \sum_{n=1}^{+\infty} nx^n = x\sum_{n=1}^{+\infty} nx^{n-1} = x\sum_{n=1}^{+\infty} \frac{d}{dx}\left(x^n\right) = x\frac{d}{dx}\left(\sum_{n=1}^{+\infty} x^n\right) = x\frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{x}{(1-x)^2}$$

   Hence $\displaystyle\sum_{n=1}^{+\infty} \frac{n}{3^n} = S_5\left(\frac{1}{3}\right) = \frac{3}{4}$

6. $\displaystyle\sum_{n=1}^{+\infty} \frac{n^2}{3^n}$

   For this question, it is interesting to introduce $S_6(x) = \displaystyle\sum_{n=1}^{+\infty} n^2x^n$ whose radius of convergence is 1 (check it).
   Hence for $x \in (-1, 1)$ we have:

$$\begin{aligned} S_6(x) &= \sum_{n=1}^{+\infty} n^2x^n = x\sum_{n=1}^{+\infty} n\cdot nx^{n-1} = x\sum_{n=1}^{+\infty} n\frac{d}{dx}\left(x^n\right) = x\frac{d}{dx}\left(\sum_{n=1}^{+\infty} nx^n\right) = x\frac{d}{dx}S_5(x) \\ &= x\frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) = x\frac{(1-x)^2+2x(1-x)}{(1-x)^4} = x\frac{1-x+2x}{(1-x)^3} \\ &= \frac{x(1+x)}{(1-x)^3} \end{aligned}$$

   Hence

$$\sum_{n=1}^{+\infty} \frac{n^2}{3^n} = S_6\left(\frac{1}{3}\right) = \frac{3}{2}$$

7. $S_7(x) = \sum_{n=0}^{+\infty} \frac{x^n}{(n+2)n!}$

Notice that the radius of convergence of S_7 is $+\infty$ (check it).

Hence, for $x \in \mathbb{R} \setminus \{0\}$: (I am going to divide by x so I'll treat the case $x = 0$ after).

$$\begin{aligned} S_7(x) &= \sum_{n=0}^{+\infty} \frac{x^n}{(n+2)n!} = \frac{1}{x^2} \sum_{n=0}^{+\infty} \frac{1}{n!} \frac{x^{n+2}}{n+2} = \frac{1}{x^2} \sum_{n=0}^{+\infty} \frac{1}{n!} \int_0^x t^{n+1} dt = \frac{1}{x^2} \int_0^x t \sum_{n=0}^{+\infty} \frac{1}{n!} t^n dt \\ &= \frac{1}{x^2} \int_0^x te^t dt \\ &= \frac{1}{x^2} \left(\left[te^t \right]_0^x - \int_0^x e^t dt \right) \quad \textit{Integration by parts} \\ &= \frac{1}{x^2} \left(xe^x - e^x + 1 \right) \end{aligned}$$

And $S_7(0) = \frac{1}{2}$ (that's simply the constant term of the series: all the x^n vanishes at $x = 0$ except x^0 which is a notation for 1).

Hence $S_7(x) = \begin{cases} \frac{xe^x - e^x + 1}{x^2} & \text{if } x \neq 0 \\ \frac{1}{2} & \text{otherwise} \end{cases}$

8. $\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)3^n}$

Again, notice that $\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)3^n} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} \left(\frac{1}{\sqrt{3}} \right)^{2n}$ so we introduce $S_8(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} x^{2n}$ whose radius of convergence is 1 (check it).

Hence, for $x \in (-1, 1) \setminus \{0\}$ (I'm going to divide by x so I avoid the case $x = 0$ which is not useful to answer this question).

$$\begin{aligned} S_8(x) &= \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} x^{2n} = \frac{1}{x} \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \frac{1}{x} \sum_{n=0}^{+\infty} (-1)^n \int_0^x t^{2n} dt \\ &= \frac{1}{x} \int_0^x \sum_{n=0}^{+\infty} (-1)^n t^{2n} dt = \frac{1}{x} \int_0^x \sum_{n=0}^{+\infty} \left(-t^2 \right)^n dt = \frac{1}{x} \int_0^x \frac{1}{1+t^2} dt = \frac{\arctan x}{x} \end{aligned}$$

And $\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)3^n} = S_8 \left(\frac{1}{\sqrt{3}} \right) = \sqrt{3} \arctan \left(\frac{1}{\sqrt{3}} \right) = \sqrt{3} \frac{\pi}{6} = \frac{\pi}{2\sqrt{3}}$

9. $\displaystyle\sum_{n=0}^{+\infty}(-1)^n\frac{n+1}{(2n)!}2^n$

First notice that

$$\sum_{n=0}^{+\infty}(-1)^n\frac{n+1}{(2n)!}2^n = \sum_{n=0}^{+\infty}\frac{(-1)^n n}{(2n)!}\left(\sqrt{2}\right)^{2n} + \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\left(\sqrt{2}\right)^{2n}$$

The second sum is $\cos\left(\sqrt{2}\right)$.

The first sum is equal to $f\left(\sqrt{2}\right)$ where $f(x) = \displaystyle\sum_{n=0}^{+\infty}\frac{(-1)^n n}{(2n)!}x^{2n}$ has a radius of convergence equal to $+\infty$ (check it).

Then for $x \in \mathbb{R}$, (*notice that for $n = 0$, the coefficient is zero*)

$$\begin{aligned} f(x) &= \sum_{n=0}^{+\infty}\frac{(-1)^n n}{(2n)!}x^{2n} = \frac{x}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}2nx^{2n-1} = \frac{x}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\frac{d}{dx}\left(x^{2n}\right) \\ &= \frac{x}{2}\frac{d}{dx}\left(\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}x^{2n}\right) = \frac{x}{2}\frac{d}{dx}(\cos(x)) = -\frac{x\sin x}{2} \end{aligned}$$

Then $\displaystyle\sum_{n=0}^{+\infty}\frac{(-1)^n n}{(2n)!}\left(\sqrt{2}\right)^{2n} = f\left(\sqrt{2}\right) = -\frac{\sqrt{2}}{2}\sin\left(\sqrt{2}\right)$.

And

$$\sum_{n=0}^{+\infty}(-1)^n\frac{n+1}{(2n)!}2^n = \cos\left(\sqrt{2}\right) - \frac{\sqrt{2}}{2}\sin\left(\sqrt{2}\right)$$

Notice that if you didn't think about the $\sqrt{2}$ that's not a big deal here, indeed:

$$\sum_{n=0}^{+\infty}\frac{(-1)^n n}{(2n)!}2^n = g(2)$$

where, if $x > 0$,

$$\begin{aligned} g(x) &= \sum_{n=0}^{+\infty}\frac{(-1)^n n}{(2n)!}x^n \\ &= x\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}nx^{n-1} \\ &= x\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\frac{d}{dx}\left(x^n\right) \\ &= x\frac{d}{dx}\left(\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}x^n\right) \\ &= x\frac{d}{dx}\left(\cos\left(\sqrt{x}\right)\right) \\ &= -\frac{x\sin\left(\sqrt{x}\right)}{2\sqrt{x}} \end{aligned}$$