Power series and Taylor series: Applications

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Other operations with Taylor series

Obtain the **terms of degree less than or equal to 4** of the Maclaurin series of these functions:

1. $f(x) = e^x \sin x$

2. $g(x) = e^{\sin x}$

Hint: Treat the power series the same way you would treat a polynomial.

Follow-up questions:
Compute $g^{(3)}(0)$ and $g^{(4)}(0)$.

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Add these series

1. $$\sum_{n=2}^{\infty} \frac{(-2)^n}{(2n+1)!}$$ *Hint:* Think of $\sin$

2. $$\sum_{n=0}^{\infty} (4n+1)x^{4n+2}$$ *Hint:* $\frac{d}{dx}\left[x^{4n+1}\right] = ???$

3. $$\sum_{n=0}^{\infty} \frac{2^n}{(2n)!}$$ *Hint:* Combine e^x and e^{-x}

4. $$\sum_{n=2}^{\infty} \frac{1}{n(n-1)} x^n$$ With two methods!

Limits

Compute these limits by writing out the first few terms of the Maclaurin series of numerator and denominator:

1. $$\lim_{x\to 0} \frac{\sin x - x}{x^3}$$
2. $$\lim_{x\to 0} \frac{6\sin x - 6x + x^3}{x^5}$$
3. $$\lim_{x\to 0} \frac{x^2 \sin x^2 - x^4}{e^{x^8} - 1}$$
4. Find a value of $a \in \mathbb{R}$ such that the limit

$$\lim_{x\to 0} \frac{e^{\sin x} - e^x + ax^3}{x^4}$$

exists and is not 0. Then compute the limit.

Consider the function

$$F(x) = \int_0^x \frac{\sin t}{t}\, dt.$$

It is not possible to find an elementary antiderivative.

1. Write $F(x)$ as a power series.
2. Estimate $F(1)$ with an error smaller than 0.01.

Add these series:

5. $$\sum_{n=1}^{\infty} \frac{n}{3^n}$$

6. $$\sum_{n=1}^{\infty} \frac{n^2}{3^n}$$

7. $$\sum_{n=0}^{\infty} \frac{x^n}{(n+2)n!}$$

8. $$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$$

9. $$\sum_{n=0}^{\infty} (-1)^n \frac{n+1}{(2n)!} 2^n$$

Hint: Take derivatives or antiderivative of series whose value you know.

More power series

For each of the following power series:

a Determine the radius of convergence, and,

b write them in terms of usual functions when x is in the interior of the interval of convergence.

1. $$\sum_{n=0}^{+\infty}(-1)^{n+1}nx^{2n+1}$$
2. $$\sum_{n=1}^{+\infty}\frac{(-1)^n}{4n}x^{4n-1}$$
3. $$\sum_{n=0}^{+\infty}(n^2+1)2^{n+1}x^n$$

Parity

1. Let f be an odd C^∞ function. What can you say about its Maclaurin series? What if f is even?

   Hint: Think of $\sin$ and $\cos$.

2. Prove it.

   Hint: Use the general formula for the Maclaurin series. What can you say about $h(0)$ if h is odd? If h is even?

Consider the function $F(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$

1. Prove that, for every $n \in \mathbb{N}$, $\lim_{t \to +\infty} t^n e^{-t} = 0$.
2. Prove that, for every $n \in \mathbb{N}$, $\lim_{x \to 0^+} \frac{e^{-1/x}}{x^n} = 0$.
3. Compute $F'(x)$ for $x > 0$.
4. Compute $F'(x)$ for $x < 0$.
5. Compute $F'(0)$ from the definition.
6. Compute $F''(0)$ from the definition.
7. Prove that for every $n \in \mathbb{N}$, $F^{(n)}(0) = 0$.
8. Write the Maclaurin series for F at 0.
9. Is F analytic? Is F C^∞?

A C^∞ but not analytic function

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An interesting power series[1]

Denote by a_n the n-th decimal digit of π.

Define the power series $S(x) = \sum_{n=1}^{\infty} a_n x^n$.

Find the radius of convergence and the interval of convergence of S.

Hint:

3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534

So $a_1 = 1,\ a_2 = 4,\ a_3 = 1,\ a_4 = 5 \ldots$

[1] From the lecture of March 25.