Power series and Taylor series: Applications

UNIVERSITY OF TORONTO

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For Wednesday, watch the videos: 14.11, 14.13, 14.15

Some digits of pi in English:

How I wish I could recollect, of circle round, the exact relation Archimede learned.

And in French:

Que j'aime à faire apprendre un nombre utile aux sages !
Immortel Archimède, artiste ingénieur,
Qui de ton jugement peut priser la valeur ?
Pour moi, ton problème eut de pareils avantages.
Jadis, mystérieux, un problème bloquait
Tout l'admirable procédé, l'œuvre grandiose
Que Pythagore découvrit aux anciens Grecs.
Ô quadrature ! vieux tourment du philosophe !
Insoluble rondeur, trop longtemps vous avez
Défié Pythagore et ses imitateurs.

3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706 79

Recap 1: Power series

1. Given a power series $S(x) = \sum_{n=0}^{+\infty} a_n(x-\alpha)^n$ centered at α, there exists a unique $0 \le R \le +\infty$, called the radius of convergence of S, such that $|x-\alpha| < R \implies S(x)$ ACV and $|x-\alpha| > R \implies S(x)$ DV

2. The interval of convergence is either $(\alpha - R, \alpha + R)$ or $[\alpha - R, \alpha + R)$ or $(\alpha - R, \alpha + R]$ or $[\alpha - R, \alpha + R]$.

3. $\sum_{n=0}^{+\infty} a_n(x-\alpha)^n + \sum_{n=0}^{+\infty} b_n(x-\alpha)^n = \sum_{n=0}^{+\infty} (a_n + b_n)(x-\alpha)^n$ for $|x-\alpha| < \min(R_A, R_B)$
$R_{A+B} = \min(R_A, R_B)$ when $R_A \ne R_B$ or $R_{A+B} \ge \min(R_A, R_B)$ otherwise

4. $\lambda \sum_{n=0}^{+\infty} a_n(x-\alpha)^n = \sum_{n=0}^{+\infty} (\lambda a_n)(x-\alpha)^n$ with same radius of convergence for $\lambda \ne 0$.

5. S is C^0 on $(\alpha - R, \alpha + R)$ and $\int_\alpha^x \sum_{n=0}^{+\infty} a_n(t-\alpha)^n dt = \sum_{n=0}^{+\infty} \frac{a_n}{n+1}(x-\alpha)^{n+1}$ with same radius of convergence.

6. S is differentiable on $(\alpha - R, \alpha + R)$ and $S'(x) = \sum_{n=1}^{+\infty} n a_n (x-\alpha)^{n-1}$

Recap 1: Power series

1. Given a power series $S(x) = \sum_{n=0}^{+\infty} a_n (x-\alpha)^n$ centered at α, there is a unique $0 \leq R \leq +\infty$ s.t. $|x-\alpha| < R \implies S(x)\ ACV$ and $|x-\alpha| > R \implies S(x)\ DV$

2. Interval of CV: $(\alpha - R, \alpha + R)$ or $[\alpha - R, \alpha + R)$ or $(\alpha - R, \alpha + R]$ or $[\alpha - R, \alpha + R]$.

3. $\sum_{n=0}^{+\infty} a_n(x-\alpha)^n + \sum_{n=0}^{+\infty} b_n(x-\alpha)^n = \sum_{n=0}^{+\infty} (a_n + b_n)(x-\alpha)^n$ when $|x-\alpha| < \min(R_A, R_B)$

4. $\lambda \sum_{n=0}^{+\infty} a_n(x-\alpha)^n = \sum_{n=0}^{+\infty} (\lambda a_n)(x-\alpha)^n$

5. S is C^0 on $(\alpha - R, \alpha + R)$ and $\int_\alpha^x \sum_{n=0}^{+\infty} a_n(t-\alpha)^n dt = \sum_{n=0}^{+\infty} \frac{a_n}{n+1}(x-\alpha)^{n+1}$

6. S is differentiable on $(\alpha - R, \alpha + R)$ and $S'(x) = \sum_{n=1}^{+\infty} n a_n (x-\alpha)^{n-1}$

Recap 1: Power series

1. Given a power series $S(x) = \sum_{n=0}^{+\infty} a_n (x-\alpha)^n$ centered at α, there is a unique $0 \leq R \leq +\infty$ s.t. $|x-\alpha| < R \implies S(x)\ ACV$ and $|x-\alpha| > R \implies S(x)\ DV$
   (R is called the *radius of convergence of S*)

2. Interval of CV: $(\alpha - R, \alpha + R)$ or $[\alpha - R, \alpha + R)$ or $(\alpha - R, \alpha + R]$ or $[\alpha - R, \alpha + R]$.

3. $\sum_{n=0}^{+\infty} a_n (x-\alpha)^n + \sum_{n=0}^{+\infty} b_n (x-\alpha)^n = \sum_{n=0}^{+\infty} (a_n + b_n)(x-\alpha)^n$ when $|x-\alpha| < \min(R_A, R_B)$
   (Remark: $R_{A+B} = \min(R_A, R_B)$ when $R_A \neq R_B$ or $R_{A+B} \geq \min(R_A, R_B)$ otherwise)

4. $\lambda \sum_{n=0}^{+\infty} a_n (x-\alpha)^n = \sum_{n=0}^{+\infty} (\lambda a_n)(x-\alpha)^n$ *(same radius of convergence for $\lambda \neq 0$).*

5. S is C^0 on $(\alpha - R, \alpha + R)$ and $\int_\alpha^x \sum_{n=0}^{+\infty} a_n (t-\alpha)^n dt = \sum_{n=0}^{+\infty} \frac{a_n}{n+1}(x-\alpha)^{n+1} = \sum_{n=1}^{+\infty} \frac{a_{n-1}}{n}(x-\alpha)^n$
   (with same radius of convergence).

6. S is differentiable on $(\alpha - R, \alpha + R)$ and $S'(x) = \sum_{n=1}^{+\infty} n a_n (x-\alpha)^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} (x-\alpha)^n$

Recap 2: Taylor polynomials and Taylor series

1. Let $f : I \to \mathbb{R}$ be a function defined on an interval I and a an interior point of I. Assume that f is n times differentiable at a.
   We define its *Taylor polynomial of order n at a* as

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

2. Notice that P_n is the unique polynomial of degree at most n such that

$$\lim_{x\to a} \frac{f(x) - P_n(x)}{(x-a)^n} = 0$$

3. Notice that P_n is the unique polynomial of degree at most n such that

$$P_n(a) = f(a),\ P_n'(a) = f'(a),\ P_n''(a) = f''(a), \dots,\ P_n^{(n)}(a) = f^{(n)}(a)$$

4. Let $f : I \to \mathbb{R}$ a function defined on an interval I and a an interior point of I. Assume that f is C^∞ at a then we define its *Taylor series at a* as

$$T_a(x) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Recap 3: analytic functions

1. Let I be an interval and a be an interior point of I. We say that a C^∞ function $f : I \to \mathbb{R}$ is analytic at a if there exists $r > 0$ such that
$$|x-a| < r \implies f(x) = T_a(x)$$
2. How can a C^∞ function not be analytic?
   a. The radius of convergence of $T_a(x)$ is 0, or,
   b. The radius of convergence is > 0 but $f(x) \neq T_a(x)$ for x close to a.
3. If around a a function is equal to a power series centered at a, then the latter is the Taylor series of the function at a.
   There is no other possibility and you can identify the coefficients.

   So if you can prove that $f(x) = \sum_{k=0}^{+\infty} a_k (x-a)^k$ when x is close to a then you know that $a_k = \frac{f^{(k)}(a)}{k!}$.
   That's what we used last week on Wednesday to compute the n-th derivatives in slides 5 and 6.

Recap 3: analytic functions

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$$|x - a| < r \implies f(x) = T_a(x)$$
2. How can a C^∞ function not be analytic?
   a. The radius of convergence of $T_a(x)$ is 0, or,
   b. The radius of convergence is > 0 but $f(x) \neq T_a(x)$ for x close to a.
3. If around a a function is equal to a power series centered at a, then the latter is the Taylor series of the function at a.
There is no other possibility and you can identify the coefficients.
4. *Taylor's theorem with Lagrange remainder.*
Let f be $(n+1)$ times differentiable on an interval I and $a \in I$ then $\forall x \in I \setminus \{a\}, \begin{cases} \exists \xi \in (a, x) & \text{if } a < x \\ \exists \xi \in (x, a) & \text{if } a > x \end{cases}$, such that
$$f(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$
5. Beware: ξ depends on n, x, a.

Recap 4: ex. of application of Lagrange's remainder[1]

Let $f : I \to \mathbb{R}$ be C^∞. If there exist $M \geq 0$ and $J_a = (a-r, a+r) \subset I$ s.t.

$$\forall n \in \mathbb{N},\ \forall x \in J_a,\ \left| f^{(n)}(x) \right| \leq M$$

then f is analytic at a.

Proof. Let $x \in J_a$. Let $n \in \mathbb{N}$.
By Taylor's theorem with Lagrange remainder, there exists $\xi \in J_a$ such that

$$\left| f(x) - \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) \right| = \left| \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \right| \leq M \left| \frac{(x-a)^{n+1}}{(n+1)!} \right|$$

Using the ratio test, we easily check that $\sum_{n=0}^{+\infty} \left| \frac{(x-a)^{n+1}}{(n+1)!} \right|$ is convergent.

Hence $\lim_{n \to +\infty} \left| \frac{(x-a)^{n+1}}{(n+1)!} \right| = 0$.

So $\forall x \in J_a,\ f(x) = \sum_{k=0}^{+\infty} \frac{(x-a)^k}{k!} f^{(k)}(a)$. ■

[1] That's exactly the method we used last Wednesday for Slide 3.

Recap 5: some results to know

1. $\forall x \in \mathbb{R},\ e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$ (recall that $0! = 1$)
2. $\forall x \in \mathbb{R},\ \cos(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ and $\sin(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$
3. $\forall x \in (-1, 1],\ \ln(1+x) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} x^n$
4. $\forall x \in (-1, 1),\ \frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n$
5. $\forall x \in (-1, 1),\ (1+x)^\alpha = 1 + \sum_{n=1}^{+\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n$

   (The last one holds for $x \in \mathbb{R}$ when $\alpha \in \mathbb{N}$.)

Keep in mind that power series behave well with respect to the usual operations: use them to reduce to the above results.

Greek alphabet (or “spaghetti”?)

A	α	Alpha	I	ι	Iota	P	ρ ϱ	Rho
B	β	Beta	K	κ	Kappa	Σ	σ ς	Sigma
Γ	γ	Gamma	Λ	λ	Lambda	T	τ	Tau
Δ	δ	Delta	M	μ	Mu	Υ	υ	Upsilon
E	ϵ ε	Epsilon	N	ν	Nu	Φ	ϕ φ	Phi
Z	ζ	Zeta	Ξ	ξ	Xi	X	χ	Chi
H	η	Eta	O	o	Omicron	Ψ	ψ	Psi
Θ	θ ϑ	Theta	Π	π ϖ	Pi	Ω	ω	Omega

Write the following functions as power series centered at 0.
First by using the sigma notation, and then by writing out the first few terms.

1. $f(x) = \dfrac{x^2}{1+x}$
2. $f(x) = (e^x)^2$
3. $f(x) = \sin\left(2x^3\right)$
4. $f(x) = \cos^2 x$
5. $f(x) = \ln \dfrac{1+x}{1-x}$
6. $f(x) = \dfrac{1}{(x-1)(x-2)}$
7. $f(x) = \displaystyle\int_0^x \cos(t^2)dt$

Obtain the **terms of degree less than or equal to 4** of the Maclaurin series of these functions:

1. $f(x) = e^x \sin x$

2. $g(x) = e^{\sin x}$

Hint: Treat the power series the same way you would treat a polynomial.

Follow-up questions:
Compute $g^{(3)}(0)$ and $g^{(4)}(0)$.

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