Analytic functions

March 27th, 2019

For next week

For Monday (Apr 1), watch the videos:

- Applications: 14.12, 14.14

For Wednesday (Apr 3), watch the videos:

- Applications: 14.11, 14.13, 14.15

cos is analytic

1. Recall Taylor's theorem with Lagrange remainder.

2. Prove that $\cos^{(k)}(x) = \cos\left(x + k\frac{\pi}{2}\right)$

3. Prove that

$$\forall x \in \mathbb{R},\ \cos(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

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Let f be $(n+1)$ times differentiable on an interval I and $a \in I$ then $\forall x \in I \setminus \{a\}$, $\begin{cases} \exists \xi \in (a,x) & \text{if } a < x \\ \exists \xi \in (x,a) & \text{if } a > x \end{cases}$, such that

$$f(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

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The binomial series

Let $\alpha \in \mathbb{R}$. Let $f(x) = (1+x)^\alpha$.

1. Find a formula for its derivatives $f^{(n)}(x)$.
2. Write its Maclaurin series at 0. Call it $S(x)$.
3. What is special about this series when $\alpha \in \mathbb{N}$?
4. Now assume $\alpha \notin \mathbb{N}$. Find the radius of convergence of the series $S(x)$.
5. One may prove that $f(x) = S(x)$ for $x \in (-1, 1)$. *(It is easier to use another remainder theorem or a method involving ODE rather than with Lagrange remainder, so we skip it)*

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$$g(x) = \frac{1}{\sqrt{1-x^2}}$$

as a power series centered at 0.

2. Write

$$h(x) = \arcsin x$$

as a power series centered at 0.
Hint: Compute $h'(x)$.

3. Deduce from the above a formula for $h^{(N)}(0)$.

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as a power series centered at 0.

Hint: Compute the first derivative.

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