MAT137Y1 – LEC0501 *Calculus!*

Power series and Taylor polynomials



March 20th, 2019

For Monday (Mar 25), watch the videos:

• Taylor series: 14.5, 14.6

For Wednesday (Mar 27), watch the videos:

• Analytic functions: 14.7, 14.8, 14.9, 14.10

Prove the following result:

Theorem

Let
$$S(x) = \sum_{n=0}^{+\infty} a_n x^n$$
 be a power series and $x_0 \in \mathbb{R}$.
IF $S(x_0)$ is convergent THEN $S(x)$ is absolutely convergent for any $x \in \mathbb{R}$ such that $|x| < |x_0|$.

1 Prove that
$$\lim_{n \to +\infty} a_n x_0^n = 0.$$

- **2** Prove that the sequence $(a_n x_0^n)_n$ is bounded.
- **3** For $x \in \mathbb{R}$, express $a_n x^n$ in terms of $a_n x_0^n$.
- 4 Conclude.

(your argument should rely on some geometric series)

Radius of convergence - Homework¹

Prove the following result:

Theorem

Given a power series $S(x) = \sum_{n=0}^{+\infty} a_n x^n$, there exists a unique $0 \le R \le +\infty$ such that 1 If |x| < R then S(x) is absolutely convergent. 2 If |x| > R then S(x) is divergent.

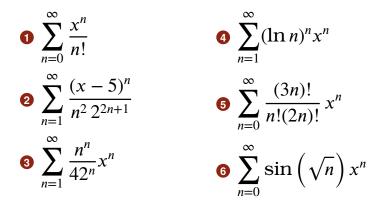
• Hint: Study the set (for a least upper bound)

 $\mathcal{E} = \{ |x|, x \in \mathbb{R}, S(x) \text{ is convergent} \}$

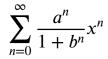
• Notice that we conclude nothing when |x| = R.

¹We may come back to this slide later.

Find the radius of convergence and the interval of convergence of each power series:



Find the radius of convergence of



Writing functions as power series

Using the geometric series, we know how to write the function $F(x) = \frac{1}{1-x}$ as a power series centered at 0:

$$F(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$
 for $|x| < 1$

Write these functions as power series centered at 0:

1
$$f(x) = \frac{1}{1+x}$$

2 $g(x) = \frac{1}{1-x^2}$
3 $h(x) = \frac{1}{2-x}$
4 $k(x) = \ln(1+x)$

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- If f and g are C^n functions, then f + g is a C^n function.
- **2** f is continuous if and only if f is C^0 .
- **3** f is differentiable if and only if f is C^1 .
- If f is a C^n function, then f is a C^{n-1} function.
- f is Cⁿ if and only if f is n times differentiable and f⁽ⁿ⁾ is continuous.

An explicit equation for Taylor polynomials

1 Find a polynomial P of degree 3 that satisfies

$$P(0) = 1$$
, $P'(0) = 5$, $P''(0) = 3$, $P'''(0) = -7$

Pind all polynomials P that satisfy

$$P(0) = 1$$
, $P'(0) = 5$, $P''(0) = 3$, $P'''(0) = -7$

- Find an explicit formula for the 3-rd Taylor polynomial for a function *f* at 0.
- Find an explicit formula for the *n*-th Taylor polynomial for a function *f* at 0.

Use the new explicit formula from the previous slide to compute the Taylor polynomials for these functions at 0:

1
$$f(x) = e^{x}$$

2 $g(x) = \sin x$
3 $h(x) = \cos x$