Power series and Taylor polynomials

March 20th, 2019

For next week

For Monday (Mar 25), watch the videos:

- Taylor series: 14.5, 14.6

For Wednesday (Mar 27), watch the videos:

- Analytic functions: 14.7, 14.8, 14.9, 14.10

Convergence of power series

Prove the following result:

Theorem

Let $S(x) = \sum_{n=0}^{+\infty} a_n x^n$ be a power series and $x_0 \in \mathbb{R}$.
IF $S(x_0)$ is convergent THEN $S(x)$ is absolutely convergent for any $x \in \mathbb{R}$ such that $|x| < |x_0|$.

1. Prove that $\lim_{n \to +\infty} a_n x_0^n = 0$.
2. Prove that the sequence $(a_n x_0^n)_n$ is bounded.
3. For $x \in \mathbb{R}$, express $a_n x^n$ in terms of $a_n x_0^n$.
4. Conclude.
   (your argument should rely on some geometric series)

Radius of convergence - *Homework*[1]

Prove the following result:

Theorem

Given a power series $S(x) = \sum_{n=0}^{+\infty} a_n x^n$, there exists a unique $0 \leq R \leq +\infty$ such that

1. If $|x| < R$ then $S(x)$ is absolutely convergent.
2. If $|x| > R$ then $S(x)$ is divergent.

- *Hint:* Study the set (for a least upper bound)

$$\mathcal{E} = \{|x|,\ x \in \mathbb{R},\ S(x) \text{ is convergent}\}$$

- Notice that we conclude nothing when $|x| = R$.

[1] We may come back to this slide later.

Radius and interval of convergence

Find the radius of convergence and the interval of convergence of each power series:

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2\, 2^{2n+1}}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{42^n} x^n$
4. $\displaystyle\sum_{n=1}^{\infty} (\ln n)^n x^n$
5. $\displaystyle\sum_{n=0}^{\infty} \frac{(3n)!}{n!(2n)!} x^n$
6. $\displaystyle\sum_{n=0}^{\infty} \sin\left(\sqrt{n}\right) x^n$

Radius of convergence - *Homework*

Find the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{a^n}{1+b^n} x^n$$

Writing functions as power series

Using the geometric series, we know how to write the function $F(x) = \dfrac{1}{1-x}$ as a power series centered at 0:

$$F(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

Write these functions as power series centered at 0:

1. $f(x) = \dfrac{1}{1+x}$
2. $g(x) = \dfrac{1}{1-x^2}$
3. $h(x) = \dfrac{1}{2-x}$
4. $k(x) = \ln(1+x)$

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True or False – C^n functions

1. If f and g are C^n functions, then $f+g$ is a C^n function.
2. f is continuous if and only if f is C^0.
3. f is differentiable if and only if f is C^1.
4. If f is a C^n function, then f is a C^{n-1} function.
5. f is C^n if and only if f is n times differentiable and $f^{(n)}$ is continuous.

An explicit equation for Taylor polynomials

1. Find a polynomial P of degree 3 that satisfies

$$P(0) = 1, \quad P'(0) = 5, \quad P''(0) = 3, \quad P'''(0) = -7$$

2. Find *all* polynomials P that satisfy

$$P(0) = 1, \quad P'(0) = 5, \quad P''(0) = 3, \quad P'''(0) = -7$$

3. Find an explicit formula for the 3-rd Taylor polynomial for a function f at 0.
4. Find an explicit formula for the n-th Taylor polynomial for a function f at 0.

Examples - *Homework*

Use the new explicit formula from the previous slide to compute the Taylor polynomials for these functions at 0:

1. $f(x) = e^x$
2. $g(x) = \sin x$
3. $h(x) = \cos x$