Ratio test

March 18th, 2019

For next lecture

For Wednesday (Mar 20), watch the videos:

- Power series: 14.1, 14.2
- Taylor polynomials: 14.3, 14.4

Ratio test: Convergent or divergent?

Use Ratio test to decide which series are convergent:

1. $$\sum_{n=1}^{\infty} \frac{3^n}{n!}$$
2. $$\sum_{n=1}^{\infty} \frac{(2n)!}{n!^2\ 3^{n+1}}$$
3. $$\sum_{n=1}^{\infty} \frac{n!(2n)!}{(3n)!}$$
4. $$\sum_{n=2}^{\infty} \frac{n!}{n^n}$$

Ratio test

Try to apply the ratio test to :

1. $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$
2. $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$
3. $$\sum_{n=1}^{\infty} \frac{1}{n}$$

Review: Convergent or Divergent?

1. $$\sum_{n=2}^{+\infty} \frac{\cos(n\pi)}{\ln n}$$

2. $$\sum_{n=2}^{+\infty} \frac{\sin(n\pi)}{\ln n}$$

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Conclusion: Think first, then compute!

For which $a, b \in \mathbb{R}$ is the following series convergent:

$$\sum_{n=2}^{+\infty} \frac{1}{n^a \ln(n)^b}$$

Convergent alternating series failing the AST
Homework

Consider $S = \displaystyle\sum_{n=2}^{+\infty} \frac{(-1)^n}{n+(-1)^n}$.

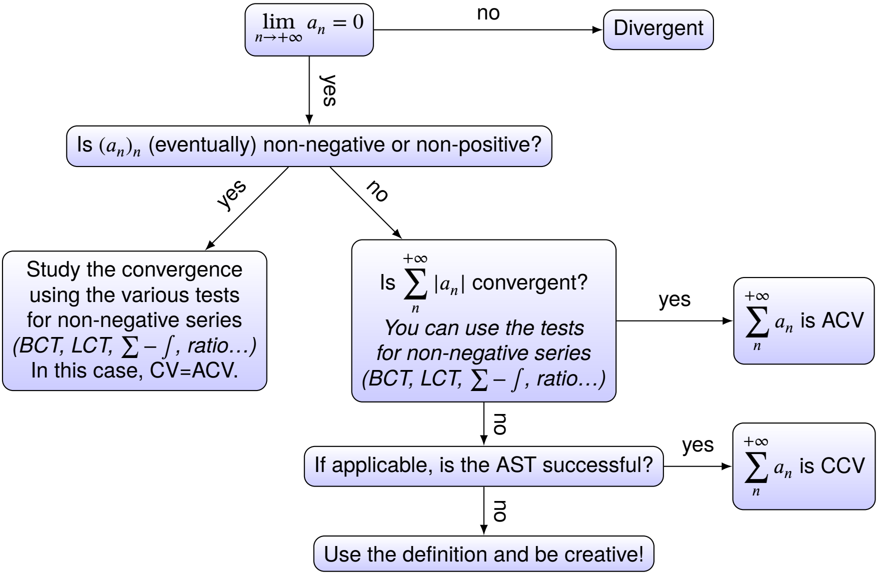
1. Why does S fail the AST?
   Hint: compare b_{2n} and b_{2n+1}.

2. Prove that S is nevertheless convergent!
   Hint: $\frac{(-1)^n}{n+(-1)^n} = \frac{(-1)^n}{n} + v_n$

Construct a series of the form $\sum_{n=1}^{\infty}(-1)^n b_n$ s.t.

- $\forall n \geq 1,\ b_n > 0$
- $\lim_{n\to\infty} b_n = 0$
- the series $\sum_{n=1}^{\infty}(-1)^n b_n$ is divergent.

Example of decision tree to study the convergence of $\sum a_n$



The two reference series

Quite often we apply the comparison tests with these two series that you MUST know!

Riemann series

$\displaystyle\sum_{n=k}^{+\infty} \frac{1}{n^p}$ is convergent if and only if $p > 1$.

Proof: use the $\sum - \int$ comparison.

Geometric series

$\displaystyle\sum_{n=k}^{+\infty} x^n$ is convergent if and only if $|x| < 1$,

then $\displaystyle\sum_{n=k}^{+\infty} x^n = \frac{x^k}{1-x}$.

Proof: use the geometric sum formula from last week.

Abel transformation (or Summation by parts)

Let $(a_n)_{n\geq p}$ and $(b_n)_{n\geq p}$ be two sequences.

We set, for $n \geq p$, $B_n = \sum_{k=p}^{n} b_k$.

1. Express b_n in terms of B_n.
   (you have to distinguish $n = p$ and $n > p$)

2. Prove that $\sum_{n=p}^{q} a_n b_n = a_p B_p + \sum_{n=p+1}^{q} a_n (B_n - B_{n-1})$.

3. Prove that $\sum_{n=p}^{q} a_n b_n = a_q B_q + \sum_{n=p}^{q-1} B_n (a_n - a_{n+1})$.
   This last equality is called "Abel transformation" or "Summation by parts".

Dirichlet's test - *Homework*

- Use an Abel transformation to prove the Dirichlet's test:
  Let $(a_n)_{n \geq p}$ and $(b_n)_{n \geq p}$ be two sequences.
  IF
  1. $(a_n)_n$ is monotonic,
  2. $\lim\limits_{n \to +\infty} a_n = 0$, and,
  3. the sequence of partial sums $\left(\sum\limits_{k=p}^{n} b_k \right)_n$ is bounded

  THEN $\sum\limits_{n=p}^{+\infty} a_n b_n$ is convergent.
- *Hint:* $\lim\limits_{q \to +\infty} a_q B_q = 0$ and the series $\sum\limits_{n=p}^{+\infty} B_n (a_n - a_{n+1})$ is absolutely convergent (hint: it is almost telescopic).
- *Application:* use this test to obtain a new proof of the AST.

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A difficult series - *Homework*

We want to prove that $\displaystyle\sum_{n=1}^{+\infty} \frac{\sin n}{n}$ is convergent.

1. Prove that
$$2\sin(n)\sin\left(\frac{1}{2}\right) = \cos\left(n-\frac{1}{2}\right) - \cos\left(n+\frac{1}{2}\right)$$
2. Use the above equality to compute
$$\sum_{n=1}^{k} \sin(n)$$
3. Conclude using the Dirichlet's test.