Absolute convergence & alternating series

March 13th, 2019

For next week

For Monday (Mar 18), watch the videos:

- Ratio test: 13.18, 13.19

For Wednesday (Mar 20), watch the videos:

- Power series: 14.1, 14.2
- Taylor polynomials: 14.3, 14.4

Convergent or divergent?
Practice of comparison tests from last Wednesday

1. $\displaystyle\sum_n^\infty \frac{n^{10}+17n^7+3}{n^{11}}$
2. $\displaystyle\sum_n^\infty \frac{\sqrt[3]{n^2+1}+1}{\sqrt{n^4+n}+n+1}$
3. $\displaystyle\sum_n^\infty \frac{2^n-40}{3^n-20}$
4. $\displaystyle\sum_n^\infty \frac{(\ln n)^{20}}{n^2}$
5. $\displaystyle\sum_n^\infty \sin^2\frac{1}{n}$
6. $\displaystyle\sum_n^\infty \frac{1}{n(\ln n)^3}$
7. $\displaystyle\sum_n^\infty \frac{1}{n\ln n}$
8. $\displaystyle\sum_n^\infty e^{-n^2}$

True or False

1. IF $(a_n)_{n\geq1}$ is convergent, THEN $(|a_n|)_{n\geq1}$ is convergent.
2. IF $(|a_n|)_{n\geq1}$ is convergent, THEN $(a_n)_{n\geq1}$ is convergent.
3. IF $\sum_{n=1}^{\infty} a_n$ is convergent, THEN $\sum_{n=1}^{\infty} |a_n|$ is convergent.
4. IF $\sum_{n=1}^{\infty} |a_n|$ is convergent, THEN $\sum_{n=1}^{\infty} a_n$ is convergent.

Convergent or divergent?

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

An AST example

Verify carefully the 3 assumptions of the Alternating Series Test for the following series.

$$\sum_{n=0}^{\infty}(-1)^n\frac{n-\pi}{e^n}$$

Can we conclude it is convergent?

Estimation

Estimate the sum

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

with an error smaller than 0.001.

Write your final answer as a rational number.

Convergent or divergent?

1. $$\sum_{n=0}^{\infty} (1.1)^n$$
2. $$\sum_{n=0}^{\infty} (0.9)^n$$
3. $$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$$
4. $$\sum_{n=1}^{\infty} \frac{(-1)^n}{e^{1/n}}$$
5. $$\sum_{n=1}^{\infty} \frac{\sin(n)}{e^n}$$
6. $$\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$$
7. $$\sum_{n=1}^{\infty} \frac{1}{n^{0.9}}$$
8. $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1.1}}$$
9. $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{0.9}}$$
10. $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$$
11. $$\sum_{n=2}^{\infty} \frac{1}{n^\alpha (\ln n)^\beta}$$
(Bertrand series)

Grouping terms - *(not mandatory homework)*

1. In general, you can **NOT** compute a series by grouping its terms. *Find a counter-example!*
2. Nevertheless, we have the following theorem: *(Prove it)*

Theorem

Let $S = \sum_{n=0}^{\infty} a_n$ be a series and $(p_n)_{n\geq 1}$ an increasing sequence of natural numbers with $p_0 = 0$.

Let $S' = \sum_{n=0}^{\infty} \left(\sum_{r=p_n}^{p_{n+1}-1} a_r \right)$.

IF

1. $\lim_{n\to+\infty} a_n = 0$
2. The sequence $r_n = p_{n+1} - p_n$ is bounded.

THEN S and S' are either both divergent or both convergent to the same value.

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Writing other sums in terms of harmonic sums - *Homework*

Recall that $H_N := \sum_{n=1}^{N} \frac{1}{n}$. It is called a harmonic sum.

Write the following sums in terms of harmonic sums.

1. $E_N = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots + \frac{1}{2N}$
2. $O_N = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2N-1}$
3. $A_{2N} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots - \frac{1}{2N}$

Compute the following series - *Homework*

Recall that:

- Last Monday, you proved there exists a convergent sequence $(c_N)_{N\geq 1}$ such that $H_N = \ln N + c_N$
- You wrote $A_{2N} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots - \frac{1}{2N}$ in terms of harmonic sums.

Use the above, to compute

$$A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \cdots$$

Challenge! Compute

$$B = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \cdots$$

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Challenge! Compute

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Answers

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \cdots = \ln 2$$

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \cdots = \frac{3}{2} \ln 2$$

Beware

When a series is conditionally convergent, you can't compute its value by rearranging the terms!
Even worse: up to picking a well-chosen rearrangement, you can obtain a series whose limit is any real number, $+\infty$, or $-\infty$ *(Riemann rearrangement theorem, Video 13.17)*.