

ABSOLUTE CONVERGENCE
& ALTERNATING SERIES

March 13th, 2019

Convergent or divergent?

Practice of comparison tests from last Wednesday

$$1 \quad \sum_n \frac{n^{10} + 17n^7 + 3}{n^{11}}$$

$$5 \quad \sum_n \sin^2 \frac{1}{n}$$

$$2 \quad \sum_n \frac{\sqrt[3]{n^2 + 1} + 1}{\sqrt{n^4 + n} + n + 1}$$

$$6 \quad \sum_n \frac{1}{n(\ln n)^3}$$

$$3 \quad \sum_n \frac{2^n - 40}{3^n - 20}$$

$$7 \quad \sum_n \frac{1}{n \ln n}$$

$$4 \quad \sum_n \frac{(\ln n)^{20}}{n^2}$$

$$8 \quad \sum_n e^{-n^2}$$

For next week

For Monday (Mar 18), watch the videos:

- Ratio test: 13.18, 13.19

For Wednesday (Mar 20), watch the videos:

- Power series: 14.1, 14.2
- Taylor polynomials: 14.3, 14.4

True or False

1 IF $(a_n)_{n \geq 1}$ is convergent, THEN $(|a_n|)_{n \geq 1}$ is convergent.

2 IF $(|a_n|)_{n \geq 1}$ is convergent, THEN $(a_n)_{n \geq 1}$ is convergent.

3 IF $\sum_{n=1}^{\infty} a_n$ is convergent, THEN $\sum_{n=1}^{\infty} |a_n|$ is convergent.

4 IF $\sum_{n=1}^{\infty} |a_n|$ is convergent, THEN $\sum_{n=1}^{\infty} a_n$ is convergent.

Convergent or divergent?

1 $\sum_{n=1}^{\infty} \frac{1}{n}$

3 $\sum_{n=1}^{\infty} \frac{1}{n^2}$

2 $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

4 $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

An AST example

Verify carefully the 3 assumptions of the Alternating Series Test for the following series.

$$\sum_{n=0}^{\infty} (-1)^n \frac{n - \pi}{e^n}$$

Can we conclude it is convergent?

Estimation

Estimate the sum

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

with an error smaller than 0.001.

Write your final answer as a rational number.

Convergent or divergent?

1 $\sum_{n=0}^{\infty} (1.1)^n$

5 $\sum_{n=1}^{\infty} \frac{\sin(n)}{e^n}$

9 $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{0.9}}$

2 $\sum_{n=0}^{\infty} (0.9)^n$

6 $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$

10 $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$

3 $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

7 $\sum_{n=1}^{\infty} \frac{1}{n^{0.9}}$

11 $\sum_{n=2}^{\infty} \frac{1}{n^\alpha (\ln n)^\beta}$
(Bertrand series)

4 $\sum_{n=1}^{\infty} \frac{(-1)^n}{e^{1/n}}$

8 $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1.1}}$

Grouping terms - (not mandatory homework)

- In general, you can **NOT** compute a series by grouping its terms. *Find a counter-example!*
- Nevertheless, we have the following theorem: (*Prove it*)

Theorem

Let $S = \sum_{n=0}^{\infty} a_n$ be a series and $(p_n)_{n \geq 1}$ an increasing sequence of natural numbers with $p_0 = 0$.

Let $S' = \sum_{n=0}^{\infty} \left(\sum_{r=p_n}^{p_{n+1}-1} a_r \right)$.

- IF
- $\lim_{n \rightarrow +\infty} a_n = 0$
 - The sequence $r_n = p_{n+1} - p_n$ is bounded.

THEN S and S' are either both divergent or both convergent to the same value.

Writing other sums in terms of harmonic sums - Homework

Recall that $H_N := \sum_{n=1}^N \frac{1}{n}$. It is called a harmonic sum.

Write the following sums in terms of harmonic sums.

$$\textcircled{1} E_N = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots + \frac{1}{2N}$$

$$\textcircled{2} O_N = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{2N-1}$$

$$\textcircled{3} A_{2N} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots - \frac{1}{2N}$$

Compute the following series - Homework

Recall that:

- Last Monday, you proved there exists a convergent sequence $(c_N)_{N \geq 1}$ such that $H_N = \ln N + c_N$
- You wrote $A_{2N} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots - \frac{1}{2N}$ in terms of harmonic sums.

Use the above, to compute

$$A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \dots$$

Challenge! Compute

$$B = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \dots$$

Answers

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \dots = \ln 2$$

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \dots = \frac{3}{2} \ln 2$$

Beware

When a series is conditionally convergent, you can't compute its value by rearranging the terms!

Even worse: up to picking a well-chosen rearrangement, you can obtain a series whose limit is any real number, $+\infty$, or $-\infty$ (*Riemann rearrangement theorem, Video 13.17*).