Integral and comparison tests

March 11$^{\text{th}}$, 2019

For next lecture

For Wednesday (Mar 13), watch the videos:

- Alternating series: 13.13, 13.14
- Absolute convergence: 13.15, (13.16), (13.17)

Rapid questions

For which values of $p \in \mathbb{R}$ are these series convergent?

1. $$\sum_{n=1}^{\infty} \frac{1}{p^n}$$
2. $$\sum_{n=1}^{\infty} \frac{1}{n^p}$$
3. $$\sum_{n=1}^{\infty} p^n$$
4. $$\sum_{n=1}^{\infty} n^p$$

From a past final exam

We know

- $\forall n \in \mathbb{N},\ 0 < a_n < 1.$
- the series $\sum_{n}^{\infty} a_n$ is convergent

Determine whether the following series are convergent, divergent, or we do not have enough information to decide:

1. $\sum_{n}^{\infty} \sin a_n$
2. $\sum_{n}^{\infty} \cos a_n$
3. $\sum_{n}^{\infty} \sqrt{a_n}$

The Euler-Mascheroni constant

Let f be a non-negative, continuous, non-increasing function on $[1, \infty)$.

1. Sketch the area $A_n = \int_1^n f(x)dx$.
2. Draw the lower sum for the partition $\{1, 2, 3, \dots, n\}$. Call it L_n.
3. Set $\mu_n = A_n - L_n$. Explain graphically why the sequence $(\mu_n)_{n\geq1}$ is monotonic and bounded. Therefore this sequence is also...?

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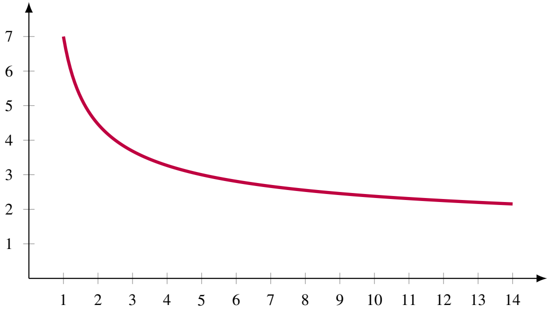

Homework: Prove question 3 formally!

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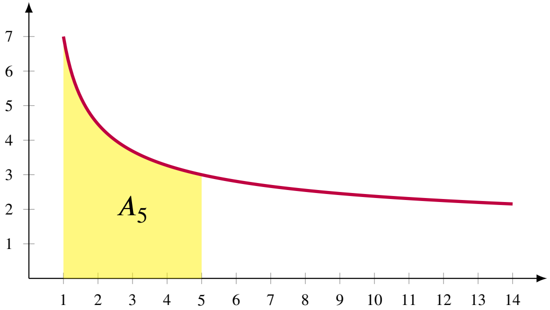

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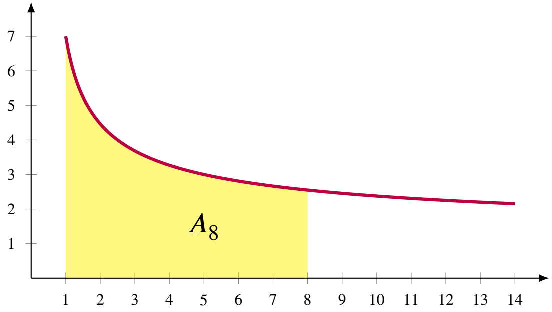

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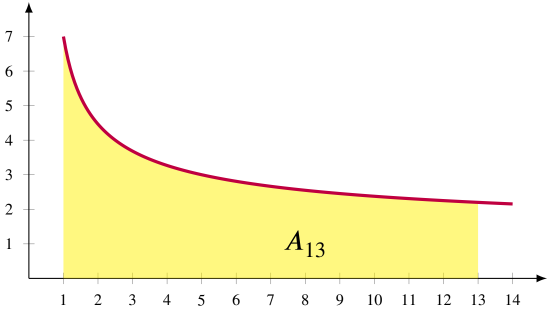

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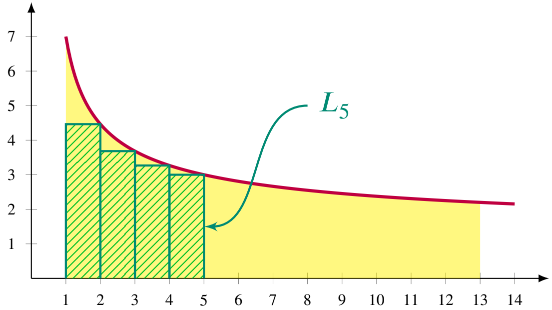

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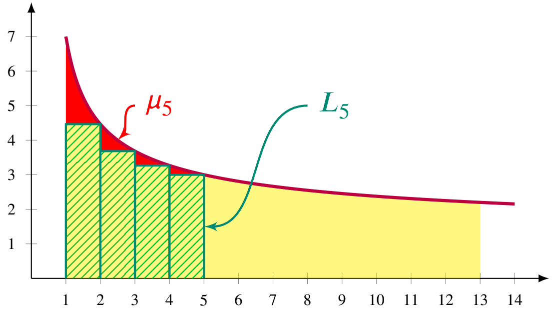

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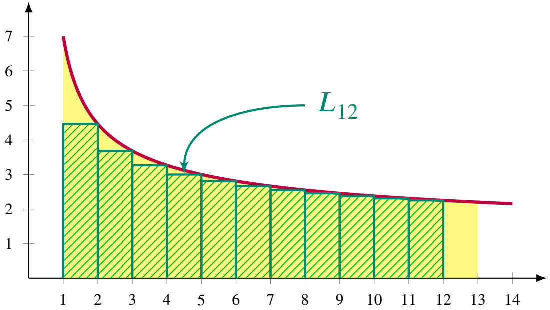

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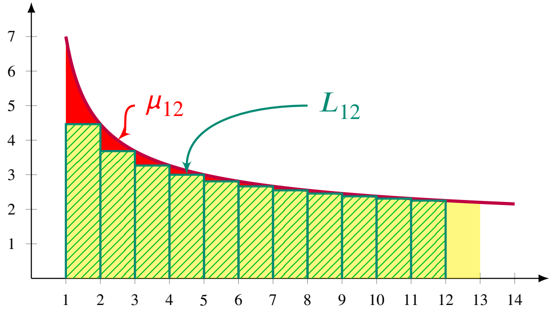

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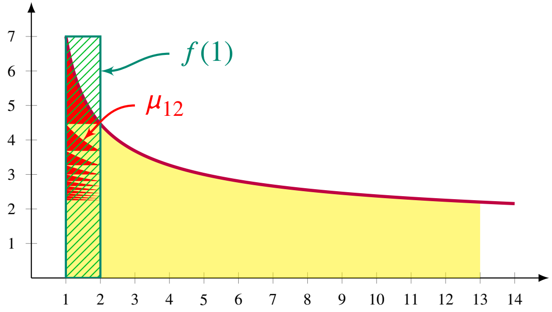

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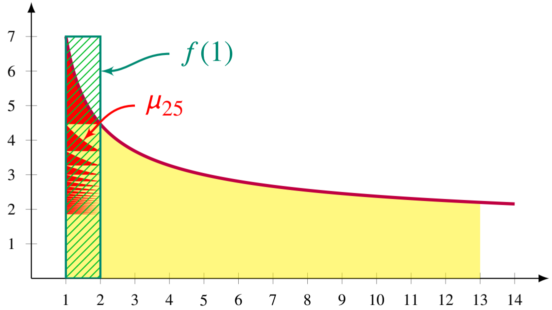

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4. Use the above result on the function $f(x) = \frac{1}{x}$ to prove the following:

   Theorem *There exists a convergent sequence* $(c_n)_{n=1}^{\infty}$ *such that, for every* $n \in \mathbb{N}$, $H_n := \sum_{k=1}^{n} \frac{1}{k} = \ln n + c_n$

5. In particular, this implies that $\boxed{\lim_{n\to\infty} \frac{H_n}{\ln n} = 1}$ and that,

   $\boxed{\text{for large } n,\ H_n \approx \ln n + \gamma}$, where $\gamma := \lim_{n\to\infty} c_n \simeq 0.5772156649\dots$ is the Euler-Mascheroni constant.