Properties of series

March 6th, 2019

For next week

For Monday (Mar 11), watch the videos:

- Integral and comparison tests: 13.10, 13.11, 13.12

For Wednesday (Mar 13), watch the videos:

- Alternating series: 13.13, 13.14
- Absolute convergence: 13.15, (13.16), (13.17)

Comment: I wrote some notes about slide 7 from last lecture.
`http://www.math.toronto.edu/campesat/ens/1819/lec39-notes.pdf`

What is wrong with this calculation? Fix it

Claim:

$$\sum_{n=2}^{\infty} \ln \frac{n}{n+1} = \ln 2$$

"Justification"

$$\begin{aligned}
\sum_{n=2}^{\infty} \ln \frac{n}{n+1} &= \sum_{n=2}^{\infty} \left[\ln n - \ln(n+1)\right] \\
&= \sum_{n=2}^{\infty} \ln(n) \; - \; \sum_{n=2}^{\infty} \ln(n+1) \\
&= (\ln 2 + \ln 3 + \ln 4 + \cdots) \; - \; (\ln 3 + \ln 4 + \cdots) \\
&= \ln 2
\end{aligned}$$

Geometric sums

Prove the following claim.

Let $x \in \mathbb{R} \setminus \{1\}$ then

$$\sum_{n=p}^{q} x^n = \frac{1 - x^{q-p+1}}{1 - x} x^p$$

Geometric sums

Prove the following claim.

Let $x \in \mathbb{R} \setminus \{1\}$ then

$$\sum_{n=p}^{q} x^n = \frac{1 - x^{q-p+1}}{1 - x} x^p$$

First term of the sum

Number of elements in the sum

Geometric series

Compute the value of the following series:

1. $S_1 = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$
2. $S_2 = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \cdots$
3. $S_3 = \frac{3}{2} - \frac{9}{4} + \frac{27}{8} - \frac{81}{16} + \cdots$
4. $S_4 = 1 + \frac{1}{2^{0.5}} + \frac{1}{2} + \frac{1}{2^{1.5}} + \frac{1}{2^2} + \frac{1}{2^{2.5}} + \cdots$
5. $S_5 = \sum_{n=1}^{\infty} (-1)^n \frac{3^n}{2^{2n+1}}$
6. $S_6 = \sum_{n=r}^{\infty} x^n$

Convergent or divergent?

1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n/2}}$
4. $\displaystyle\sum_{n=5}^{\infty} \frac{3^n}{2^{2n+1}}$
5. $\displaystyle\sum_{n=3}^{\infty} \frac{3^n}{1000 \cdot 2^{n+2}}$
6. $\displaystyle\sum_{n=0}^{\infty} (-1)^n$

Convergent or divergent?

1. $$\sum_{n=0}^{\infty} \frac{1}{2^n}$$
2. $$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$$
3. $$\sum_{n=1}^{\infty} \frac{1}{2^{n/2}}$$
4. $$\sum_{n=5}^{\infty} \frac{3^n}{2^{2n+1}}$$
5. $$\sum_{n=3}^{\infty} \frac{3^n}{1000 \cdot 2^{n+2}}$$
6. $$\sum_{n=0}^{\infty} (-1)^n$$

True or False – Series

Let $\sum_{n=0}^{\infty} a_n$ be a series. Let $\{S_n\}_{n=0}^{\infty}$ be its partial-sum sequence.

1. IF the series $\sum_{n=0}^{\infty} a_n$ is divergent, THEN $\exists n \in \mathbb{N}$ such that $a_n > 100$
2. IF the series $\sum_{n=0}^{\infty} a_n$ is divergent, THEN $\exists n \in \mathbb{N}$ such that $S_n > 100$
3. IF the series $\sum_{n=0}^{\infty} a_n$ converges

   THEN the series $\sum_{n=100}^{\infty} a_n$ converges to a smaller number.
4. IF the series $\sum_{n=0}^{\infty} a_n$ converges

   THEN the sequence $\{S_n\}_{n=0}^{\infty}$ is eventually monotonic.

True or False – Series

Let $\sum_{n=0}^{\infty} a_n$ be a series. Let $\{S_n\}_{n=0}^{\infty}$ be its partial-sum sequence.

5. IF the sequence $\{S_n\}_{n=0}^{\infty}$ is bounded and eventually monotonic, THEN the series $\sum_{n=0}^{\infty} a_n$ is convergent.

6. IF the sequence $\{S_n\}_{n=0}^{\infty}$ is increasing, THEN $\forall n > 0,\ a_n > 0$.

7. IF the sequence $\{S_n\}_{n=0}^{\infty}$ is increasing, THEN $\forall n \geq 0,\ a_n > 0$.

True or False – Series

8. IF $\lim_{n\to\infty} a_n = 0$, THEN $\sum_n^{\infty} a_n$ is convergent.

9. IF $\lim_{n\to\infty} a_n \neq 0$, THEN $\sum_n^{\infty} a_n$ is divergent.

10. IF $\sum_n^{\infty} a_n$ is convergent THEN $\lim_{n\to\infty} a_n = 0$.

11. IF $\sum_n^{\infty} a_n$ is divergent THEN $\lim_{n\to\infty} a_n \neq 0$.

Proving a property about series from the definition

Let $\sum_{n=0}^{\infty} a_n$ be a series and $c \in \mathbb{R}$.

Prove that

IF $\sum_{n=0}^{\infty} a_n$ is convergent,

THEN $\sum_{n=0}^{\infty} (ca_n)$ is also convergent and $\sum_{n=0}^{\infty} (ca_n) = c\left(\sum_{n=0}^{\infty} a_n\right)$.

Is $0.999999\ldots = 1$?

First, we need to fix the meaning of $0.999999\ldots$.

1. Write the number $0.9999999\ldots$ as a series
   Hint: $427 = 400 + 20 + 7$.

2. Compute the first few partial sums

3. Add up the series.
   Hint: it is geometric.

Is $0.999999\ldots = 1$?

First, we need to fix the meaning of $0.999999\ldots$.

1. Write the number $0.9999999\ldots$ as a series
   Hint: $427 = 400 + 20 + 7$.

2. Compute the first few partial sums

3. Add up the series.
   Hint: it is geometric.

Is $0.999999\ldots = 1$?

First, we need to fix the meaning of $0.999999\ldots$.

1. Write the number $0.9999999\ldots$ as a series
   Hint: $427 = 400 + 20 + 7$.

2. Compute the first few partial sums

3. Add up the series.
   Hint: it is geometric.

Are all decimal expansions well-defined?

The decimal expansion of a number is a series:

$$0.a_1a_2a_3a_4a_5\ldots \;=\; \frac{a_1}{10}+\frac{a_2}{100}+\frac{a_3}{1000}+\cdots$$

for any “digits” $a_1, a_2, a_3, \ldots$

Are such series always convergent, no matter which infinite string of digits we choose?

Yes! Prove it.
(Hint: all the terms in the series are positive.)

Challenge

We want to calculate the value of

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,3^n}$$

Hints:

1. Compute $\sum_{n=0}^{\infty} (-1)^n x^{2n}$
2. Compute $\frac{d}{dx}[\arctan x]$
3. Pretend you can take derivatives and antiderivatives of series the way you can take them of sums. Which series adds up to $\arctan x$?
4. Now attempt the original problem.

A more difficult challenge - part 1
Homework, not mandatory

Let $x \in (0,1)$. Then, there exists a unique sequence $(a_n)_{n\geq 1}$ of digits not eventually constant to 9 such that $x = 0.a_1a_2a_3\ldots$.
We want to prove that x is rational if and only if the sequence $(a_n)_n$ is eventually periodic, i.e.

$$\exists n_0 \geq 1,\ \exists p \geq 1,\ \forall n \geq 1,\ \left(n \geq n_0 \implies a_{n+p} = a_n\right)$$

1. Prove that if the sequence $(a_n)_n$ is eventually periodic then

$$x = r + \frac{y}{10^{n_0}} \sum_{l=0}^{+\infty} 10^{-lp}$$

where $y = a_{n_0} + \frac{a_{n_0+1}}{10} + \cdots + \frac{a_{n_0+p-1}}{10^{p-1}}$ and $r \in \mathbb{R}$.
Conclude that $x \in \mathbb{Q}$.

A more difficult challenge - part 2
Homework, not mandatory

2 Conversely, we now assume that $x = \frac{a}{b}$.

1. Prove there exist $0 < s < t$ such that the divisions of $10^s a$ and $10^t a$ by b have the same remainder.

2. We denote by $(b_n)_{n\geq 1}$ and $(c_n)_{n\geq 1}$ the digits of the fractional parts of $\frac{10^s a}{b}$ and $\frac{10^t a}{b}$.
Express b_n and c_n in terms of a_n.

3. Prove that $\frac{10^s a}{b} - \frac{10^t a}{b} \in \mathbb{Z}$.

4. Using the above question, find a relation between b_n and c_n.

5. Conclude that $(a_n)_n$ is eventually periodic.