University of Toronto – MAT137Y1 – LEC0501

Calculus!

Notes about slide 7

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For each $n \in \mathbb{N}_{>0}$, we define

$$r_n = \text{ the smallest power of 2 that is greater than or equal to } n$$

Consider the series $S = \sum_{n=1}^{\infty} \frac{1}{r_n}$.

1. Compute r_1 through r_8.

   $r_1 = 1, r_2 = 2, r_3 = 4, r_4 = 4, r_5 = 8, r_6 = 8, r_7 = 8, r_8 = 8.$

2. Compute the partial sums S_1, S_2, S_4, S_8 of the series S.

$$S_1 = \sum_{n=1}^{1} \frac{1}{r_n} = \frac{1}{r_1} = 1$$

$$S_2 = \sum_{n=1}^{2} \frac{1}{r_n} = \frac{1}{r_1} + \frac{1}{r_2} = 1 + \frac{1}{2} = \frac{3}{2}$$

$$S_4 = \sum_{n=1}^{4} \frac{1}{r_n} = \sum_{n=1}^{2} \frac{1}{r_n} + \sum_{n=3}^{4} \frac{1}{r_n} = S_2 + \frac{1}{r_3} + \frac{1}{r_4} = S_2 + \frac{1}{4} + \frac{1}{4} = S_2 + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = 2$$

$$S_8 = \sum_{n=1}^{8} \frac{1}{r_n} = \sum_{n=1}^{4} \frac{1}{r_n} + \sum_{n=5}^{8} \frac{1}{r_n} = S_4 + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = S_4 + \frac{1}{2} = 2 + \frac{1}{2} = \frac{5}{2}$$

It seems that $S_{2^{k+1}} = S_{2^k} + \frac{1}{2}$, allowing us to conjecture that

$$\forall k \in \mathbb{N},\ S_{2^k} = 1 + \frac{k}{2}$$

The above claim is true and will be useful for the next question, so let's prove it by induction.

Base case, for $k = 0$: $S_{2^0} = S_1 = 1 = 1 + \frac{0}{2}$.

Induction step:
We assume that $S_{2^k} = 1 + \frac{k}{2}$ for some $k \in \mathbb{N}$ and we want to show that $S_{2^{k+1}} = 1 + \frac{k+1}{2}$.
First, notice that

$$S_{2^{k+1}} = \sum_{n=1}^{2^{k+1}} \frac{1}{r_n} = \sum_{n=1}^{2^k} \frac{1}{r_n} + \sum_{n=2^k+1}^{2^{k+1}} \frac{1}{r_n} = S_{2^k} + \sum_{n=2^k+1}^{2^{k+1}} \frac{1}{r_n} = 1 + \frac{k}{2} + \sum_{n=2^k+1}^{2^{k+1}} \frac{1}{r_n}$$

where the last equality comes from the induction hypothesis.

Then, notice that the last sum has $(2^{k+1}) - (2^k + 1) + 1 = 2^{k+1} - 2^k = 2^k(2-1) = 2^k$ terms and moreover, for all $n = 2^k + 1, 2^k + 2, \dots, 2^{k+1}$, we have $2^k < n \le 2^{k+1}$, hence $r_n = 2^{k+1}$.
Therefore

$$S_{2^{k+1}} = 1 + \frac{k}{2} + 2^k \frac{1}{2^{k+1}} = 1 + \frac{k}{2} + \frac{1}{2} = 1 + \frac{k+1}{2}$$

3. Compute $S = \sum_{n=1}^{\infty} \frac{1}{r_n}$.

   We denote the k-th partial sum of S by $S_k = \sum_{n=1}^{k} \frac{1}{r_n}$.

   Let $k \in \mathbb{N}_{>0}$, then

$$S_{k+1} = \sum_{n=1}^{k+1} \frac{1}{r_n} = \left(\sum_{n=1}^{k} \frac{1}{r_n} \right) + \frac{1}{r_{k+1}} = S_k + \frac{1}{r_{k+1}} > S_k$$

   Hence the sequence $(S_k)_{k\ge1}$ is increasing.
   Therefore, either it is bounded from above and then convergent or it is not bounded from above and then it is divergent to $+\infty$.

   But we know that $\lim_{k\to+\infty} S_{2^k} = \lim_{k\to+\infty} \left(1 + \frac{k}{2}\right) = +\infty$.

   So $(S_k)_k$ is not bounded from above: for any $M \in \mathbb{R}$, there exists $l \in \mathbb{N}$ such that $S_k > M$ with $k = 2^l$.

   Hence, since $(S_k)_k$ is increasing and not bounded from above, we get that $\lim_{k\to+\infty} S_k = +\infty$.

   We proved that $\sum_{n=1}^{\infty} \frac{1}{r_n} = +\infty$.

4. Compute $H = \sum_{n=1}^{\infty} \frac{1}{n}$.

   Hint: "Compare" H and S.

   By definition of r_n, we know that, for any $n \in \mathbb{N}_{>0}$, we have $r_n \ge n > 0$ and therefore $\frac{1}{r_n} \le \frac{1}{n}$.

   Hence, for any $k \in \mathbb{N}_{>0}$, we have $\sum_{n=1}^{k} \frac{1}{r_n} \le \sum_{n=1}^{k} \frac{1}{n}$.

   Since $\lim_{k\to+\infty} \sum_{n=1}^{k} \frac{1}{r_n} = +\infty$, we get by comparison that $\lim_{k\to+\infty} \sum_{n=1}^{k} \frac{1}{n} = +\infty$.

   We proved that $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$.