University of Toronto – MAT137Y1 – LEC0501

Calculus!

Notes about today's lecture

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1 Definitions of improper integrals

Definition 1.1. Let $f : [a, +\infty) \to \mathbb{R}$ be a function which is integrable on each subinterval $[a, c] \subset [a, +\infty)$ (for instance continuous), then we set

$$\int_a^{+\infty} f(x)dx := \lim_{c \to +\infty} \int_a^c f(x)dx$$

whenever it makes sense.

Definition 1.2. Let $f : (-\infty, b] \to \mathbb{R}$ be a function which is integrable on each subinterval $[c, b] \subset (-\infty, b]$ (for instance continuous), then we set

$$\int_{-\infty}^{b} f(x)dx := \lim_{c \to -\infty} \int_c^b f(x)dx$$

whenever it makes sense.

Definition 1.3. Let $f : [a, b) \to \mathbb{R}$ be a function which is integrable on each subinterval $[a, c] \subset [a, b)$ (for instance continuous), then we set

$$\int_a^b f(x)dx := \lim_{c \to b^-} \int_a^c f(x)dx$$

whenever it makes sense.

Definition 1.4. Let $f : (a, b] \to \mathbb{R}$ be a function which is integrable on each subinterval $[c, b] \subset (a, b]$ (for instance continuous), then we set

$$\int_a^b f(x)dx := \lim_{c \to a^+} \int_c^b f(x)dx$$

whenever it makes sense.

Definition 1.5. Let $f : (a, b) \to \mathbb{R}$ be a function which is integrable on each subinterval $[c, d] \subset (a, b)$ where $a \in \mathbb{R}$ or $a = -\infty$ and $b \in \mathbb{R}$ or $b + \infty$ (for instance f is continuous).
We say that $\int_a^b f(x)dx$ is convergent if there exists $c \in (a, b)$ such that the improper integrals $\int_a^c f(x)dx$ of $f : (a, c] \to \mathbb{R}$ and $\int_c^b f(x)dx$ of $f : [c, b) \to \mathbb{R}$ are both convergent and then we set

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Remark 1.6. In the above definition, you need to study the two bounds separately!
Moreover, if $\int_a^b f(x)dx$ is convergent then it is for any $c \in (a, b)$ and its value doesn't depend on the choice of c.

2 Slide 8: the MCT for functions

Theorem 2.1. *Let $F : [a,b) \to \mathbb{R}$ be a function where either $b \in \mathbb{R}_{>a}$ or $b = +\infty$.*
If F is non-decreasing and bounded from above then $\lim\limits_{x \to b^-} F(x)$ exists (moreover this limit is equal to $\sup\{F(x),\ x \in [a,b)\}$).

Proof. The set $S = \{F(x),\ x \in [a,b)\}$ is

- not empty since it contains $F(a)$, and,
- bounded from above since F is.

Hence, by the "least upper bound principle", it admits a supremum $M = \sup(S)$, i.e. there exists $M \in \mathbb{R}$ satisfying

$$\begin{cases} \forall x \in [a,b),\ F(x) \leq M \\ \forall \varepsilon > 0,\ \exists x_0 \in [a,b),\ M - \varepsilon < F(x_0) \end{cases}$$

We can prove that $\lim\limits_{x \to b^-} F(x) = M$.
I am just doing the case where $b = +\infty$, the other case is quite similar (*do it as an exercise*).
We want to show that $\lim\limits_{x \to +\infty} F(x) = M$, i.e.

$$\forall \varepsilon > 0,\ \exists A \in \mathbb{R},\ \forall x \in [a,+\infty),\ (x > A \implies |F(x) - M| < \varepsilon)$$

- Let $\varepsilon > 0$.
- We know there exists $x_0 \in [a,+\infty)$ such that $M - \varepsilon < F(x_0)$.
- We set $A = x_0$.
- Let $x \in [a,+\infty)$ satisfying $x > A$.
- Since F is non-decreasing, we know that $M - \varepsilon < F(x_0) = F(A) \leq F(x)$.
- Hence $M - F(x) < \varepsilon$.
- But since M is an upper bound of F, we also have that $F(x) \leq M$.
- Therefore $0 \leq M - F(x) < \varepsilon$ which implies $|F(x) - M| = |M - F(x)| < \varepsilon$.
- We have well that $x > A \implies |F(x) - M| < \varepsilon$.

■

Theorem 2.2. *Let $F : [a,b) \to \mathbb{R}$ be a function where either $b \in \mathbb{R}_{>a}$ or $b = +\infty$.*
If F is non-decreasing and not bounded from above then $\lim\limits_{x \to b^-} F(x) = +\infty$.

Proof. You can adapt the answer of the second question of the Problem Set 8. ■

3 The BCT and the LCT

Theorem 3.1 (The BCT). *Let $f, g : [a, b) \to \mathbb{R}$ be two functions (either $b \in \mathbb{R}_{>a}$ or $b = +\infty$) such that*
(*i*) *f and g are integrable on any subinterval $[a, c] \subset [a, b)$ (for instance they are continuous), and,*
(*ii*) *$\forall x \in [a, b)$, $0 \leq f(x) \leq g(x)$.*
Under the above assumptions, the following statements hold:

1. *If $\int_a^b f(x)dx$ is divergent then $\int_a^b g(x)dx$ is divergent.*
2. *If $\int_a^b g(x)dx$ is convergent then $\int_a^b f(x)dx$ is convergent.*

Proof.
For $x \in [a, b)$, set $F(x) = \int_a^x f(t)dt$ (which is well-defined since f is integrable on $[a, x]$), then F is non-decreasing: if $x_1 < x_2$ then $F(x_2) - F(x_1) = \int_{x_1}^{x_2} f(t)dt \geq 0$.

Hence, according to the MCT, either F is bounded from above and then $\int_a^b f(t)dt = \lim_{x \to b^-} F(x)$ exists, or it is not bounded from above and the limit is $+\infty$.
The same result holds for $G = \int_a^x g(t)dt$.

First case: assume that $\int_a^b f(x)dx$ is divergent.
Since $f(x) \leq g(x)$, we have $\int_a^x f(t)dt \leq \int_a^x g(t)dt$.
Since the limit of the LHS of the inequality is $+\infty$ (by the above remark), then the limit of the RHS is also $+\infty$.

Second case: assume that $\int_a^b g(x)dx$ is convergent.
Therefore G is bounded from above by a number $M \in \mathbb{R}$.
Hence, for any $x \in [a, b)$, $F(x) = \int_a^x f(t)dt \leq \int_a^x g(t)dt = G(x) \leq M$.
Therefore $F(x)$ is non-decreasing and admits an upper bound.
We deduce from the MCT that $\int_a^b f(x)dx = \lim_{x \to b^-} F(x)$ is convergent. ■

Exercise 3.2. *Let $f, g : [a, b) \to \mathbb{R}$ be two functions (either $b \in \mathbb{R}_{>a}$ or $b = +\infty$) such that*
(*i*) *f and g are integrable on any subinterval $[a, c] \subset [a, b)$, and,*
(*ii*) *$\exists \alpha, \beta > 0$, $\forall x \in [a, b)$, $0 \leq \alpha f(x) \leq g(x) \leq \beta f(x)$.*
Prove that $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ are either both convergent or both divergent.

Theorem 3.3 (The LCT). *Let $f, g : [a, b) \to \mathbb{R}$ be two functions (either $b \in \mathbb{R}_{>a}$ or $b = +\infty$) such that*
(*i*) *f and g are integrable on any subinterval $[a, c] \subset [a, b)$ (for instance they are continuous),*
(*ii*) *$\forall x \in [a, b)$, $f(x) \geq 0$,*
(*iii*) *$\forall x \in [a, b)$, $g(x) > 0$, and,*
(*iv*) *$\lim_{x \to b^-} \frac{f(x)}{g(x)} = \lambda > 0$ exists and is positive.*
Then $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ are either both convergent or both divergent.

Proof. I'm explaining the case $b = +\infty$, the other case is exactly the same.
By definition of the limit (applied to $\varepsilon = \frac{\lambda}{2}$), there exists a $M \in \mathbb{R}$ such that $\forall x \in [a, b)$,

$$x > M \implies \left| \frac{f(x)}{g(x)} - \lambda \right| < \frac{\lambda}{2}$$

We may rewrite the conclusion as

$$\lambda - \frac{\lambda}{2} < \frac{f(x)}{g(x)} < \lambda + \frac{\lambda}{2}$$

which implies that

$$0 < \frac{\lambda}{2} g(x) < f(x) < \frac{3\lambda}{2} g(x)$$

The end of the proof now derives from the above exercise. ∎

Remark 3.4. The above results hold for improper integrals of functions of the form $f : (a, b] \to \mathbb{R}$ (i.e. when the integral is improper at the lower bound).
Indeed, if $0 \leq f(x) \leq g(x)$ on $(a, b]$ then $F(x) = \int_x^b f(t)dt$ and $G(x) = \int_x^b g(t)dt$ are non-increasing. Therefore either F is bounded from above and $\lim\limits_{x \to a^+} F(x)$ exists or $\lim\limits_{x \to a^+} F(x) = +\infty$ (and the same result holds for G).
Hence the above proof works as it is.

Remark 3.5. Do not forget to check (and recall) the assumptions of the above theorems before using them!

Remark 3.6. Notice that $\int_a^b f(x)dx$ is convergent if and only if $\int_a^b -f(x)dx$ is.
Hence, when you want to compare two functions that are both negative, you can multiply them by -1 and then apply the above results.

4 Slide 9: hints

(1) Notice that

$$\frac{\frac{x^3+2x+7}{x^5+11x^4+1}}{\frac{1}{x^2}} = \frac{x^5 + 2x^3 + 7x^2}{x^5 + 11x^4 + 1} \xrightarrow[x \to +\infty]{} 1$$

(2) Notice that

$$\frac{\frac{1}{\sqrt{x^2+x+1}}}{\frac{1}{x}} = \frac{x}{\sqrt{x^2 + x + 1}} \xrightarrow[x \to +\infty]{} 1$$

(3) Notice that

$$\frac{\frac{3\cos(x)}{x+\sqrt{x}}}{\frac{1}{\sqrt{x}}} = \frac{3\cos(x)}{\sqrt{x} + 1} \xrightarrow[x \to 0^+]{} 3$$

(4) Notice that

$$\frac{\cot(x)}{\frac{1}{x}} = \cos(x) \cdot \frac{x}{\sin(x)} \xrightarrow[x \to 0^+]{} 1$$

(5) Notice that

$$\frac{\frac{\sin(x)}{x^{3/2}}}{\frac{1}{\sqrt{x}}} = \frac{\sin(x)}{x} \xrightarrow[x \to 0^+]{} 1$$