BCT and LCT for improper integrals

February 27th, 2019

For next week

For Monday (Mar 4), watch the videos:

- Definition of series: (13.1), 13.2, 13.3, 13.4

For Wednesday (Mar 6), watch the videos:

- Properties of series: 13.5, 13.6, 13.7, 13.8, 13.9

Quick review: Riemann's improper integrals

For which values of $p \in \mathbb{R}$ is each of the following improper integrals convergent?

1. $\displaystyle\int_1^{+\infty} \frac{1}{x^p}\,dx$

2. $\displaystyle\int_0^{1} \frac{1}{x^p}\,dx$

3. $\displaystyle\int_0^{+\infty} \frac{1}{x^p}\,dx$

The BCT: True or False - A

Let $a \in \mathbb{R}$.
Let f and g be continuous functions on $[a, +\infty)$.
Assume that

$$\forall x \geq a,\ 0 \leq f(x) \leq g(x)$$

What can we conclude?

1. IF $\int_a^{+\infty} f(x)dx$ is convergent, THEN $\int_a^{+\infty} g(x)dx$ is convergent.
2. IF $\int_a^{+\infty} f(x)dx = +\infty$, THEN $\int_a^{+\infty} g(x)dx = +\infty$.
3. IF $\int_a^{+\infty} g(x)dx$ is convergent, THEN $\int_a^{+\infty} f(x)dx$ is convergent.
4. IF $\int_a^{+\infty} g(x)dx = +\infty$, THEN $\int_a^{+\infty} f(x)dx = +\infty$.

The BCT: True or False - B

Let $a \in \mathbb{R}$.
Let f and g be continuous functions on $[a, +\infty)$.
Assume that

$$\exists M \geq a, \forall x \geq M, 0 \leq f(x) \leq g(x)$$

What can we conclude?

1. IF $\int_a^{+\infty} f(x)dx$ is convergent, THEN $\int_a^{+\infty} g(x)dx$ is convergent.
2. IF $\int_a^{+\infty} f(x)dx = +\infty$, THEN $\int_a^{+\infty} g(x)dx = +\infty$.
3. IF $\int_a^{+\infty} g(x)dx$ is convergent, THEN $\int_a^{+\infty} f(x)dx$ is convergent.
4. IF $\int_a^{+\infty} g(x)dx = +\infty$, THEN $\int_a^{+\infty} f(x)dx = +\infty$.

The BCT: True or False - C

Let $a \in \mathbb{R}$.
Let f and g be continuous functions on $[a, +\infty)$.
Assume that

$$\forall x \geq a, \ f(x) \leq g(x)$$

What can we conclude?

1. IF $\int_a^{+\infty} f(x)dx$ is convergent, THEN $\int_a^{+\infty} g(x)dx$ is convergent.
2. IF $\int_a^{+\infty} f(x)dx = +\infty$, THEN $\int_a^{+\infty} g(x)dx = +\infty$.
3. IF $\int_a^{+\infty} g(x)dx$ is convergent, THEN $\int_a^{+\infty} f(x)dx$ is convergent.
4. IF $\int_a^{+\infty} g(x)dx = +\infty$, THEN $\int_a^{+\infty} f(x)dx = +\infty$.

A first application of the BCT

Determine whether

$$\int_1^{+\infty} \frac{1}{x+e^x}dx$$

is convergent or divergent.

What can you conclude?

Let $a \in \mathbb{R}$. Let f be a continuous, positive function on $[a, \infty)$.
In each of the following cases, what can you conclude about $\int_a^{+\infty} f(x)dx$?
Is it convergent, divergent, or we do not know?

1. $\forall b \geq a,\ \exists M \in \mathbb{R}$ s.t. $\int_a^b f(x)dx \leq M$.
2. $\exists M \in \mathbb{R}$ s.t. $\forall b \geq a,\ \int_a^b f(x)dx \leq M$.
3. $\exists M > 0$ s.t. $\forall x \geq a,\ f(x) \leq M$.
4. $\exists M > 0$ s.t. $\forall x \geq a,\ f(x) \geq M$.

BCT calculations

Use the BCT to determine whether each of the following is convergent or divergent

1. $\displaystyle\int_1^{+\infty} \frac{1+\cos^2 x}{x^{2/3}}\,dx$
2. $\displaystyle\int_1^{+\infty} \frac{1+\cos^2 x}{x^{4/3}}\,dx$
3. $\displaystyle\int_0^{+\infty} \frac{\arctan x^2}{1+e^x}\,dx$
4. $\displaystyle\int_0^{+\infty} e^{-x^2}\,dx$
5. $\displaystyle\int_2^{+\infty} \frac{(\ln x)^{10}}{x^2}\,dx$

MCT for functions - *Homework*

The proof of the BCT relies on the following version of the Monotone Convergence Theorem:

Theorem

Let $a \in \mathbb{R}$.
Let F be a function defined on $[a, \infty)$.

- IF F is increasing and bounded above,
- THEN $\lim_{x \to \infty} F(x)$ exists.

Prove it.

Convergent or divergent?

1. $$\int_1^{+\infty} \frac{x^3+2x+7}{x^5+11x^4+1}\,dx$$
2. $$\int_1^{+\infty} \frac{1}{\sqrt{x^2+x+1}}\,dx$$
3. $$\int_0^1 \frac{3\cos x}{x+\sqrt{x}}\,dx$$
4. $$\int_0^1 \cot x\,dx$$
5. $$\int_0^1 \frac{\sin x}{x^{3/2}}\,dx$$

Harder example

For which values of $a > 0$ is the improper integral

$$\int_0^{+\infty} \frac{\arctan x}{x^a}\, dx$$

convergent?