MAT137Y1 – LEC0501

Calculus!

Improper integrals

UNIVERSITY OF TORONTO

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For next lecture

For Wednesday (Feb 27), watch the videos:

- BCT: 12.7, 12.8
- LCT: 12.9, 12.10

Improper intergrals: definition – 1

Let $f : [a, +\infty) \to \mathbb{R}$ be a function which is integrable on each subinterval $[a, c] \subset [a, +\infty)$ (for instance continuous), then we set

$$\int_a^{+\infty} f(x)dx := \lim_{c \to +\infty} \int_a^c f(x)dx$$

whenever it makes sense.

Let $f : (-\infty, b] \to \mathbb{R}$ be a function which is integrable on each subinterval $[c, b] \subset (-\infty, b]$ (for instance continuous), then we set

$$\int_{-\infty}^{b} f(x)dx := \lim_{c \to -\infty} \int_c^b f(x)dx$$

whenever it makes sense.

Improper intergrals: definition – 2

Let $f : [a, b) \to \mathbb{R}$ be a function which is integrable on each subinterval $[a, c] \subset [a, b)$ (for instance continuous), then we set

$$\int_a^{b} f(x)dx := \lim_{c \to b^-} \int_a^c f(x)dx$$

whenever it makes sense.

Let $f : (a, b] \to \mathbb{R}$ be a function which is integrable on each subinterval $[c, b] \subset (a, b]$ (for instance continuous), then we set

$$\int_a^{b} f(x)dx := \lim_{c \to a^+} \int_c^b f(x)dx$$

whenever it makes sense.

Riemann integrals

- For which $p \in \mathbb{R}$ is the following improper integral convergent?

$$\int_1^{+\infty} \frac{1}{x^p} dx$$

- For which $p \in \mathbb{R}$ is the following improper integral convergent?

$$\int_0^{1} \frac{1}{x^p} dx$$

- For which $p \in \mathbb{R}$ is the following improper integral convergent?

$$\int_0^{+\infty} \frac{1}{x^p} dx$$

Antiderivatives

- Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.
  Can all the antiderivatives of f be written as

$$F(x) = \int_a^x f(t)dt$$

  where $a \in \mathbb{R}$?

- Find one function such that the above statement is true.
- Find one function such that the above statement is false.

- *(Additional question, not related to the above phenomenon)*
  Find all the antiderivatives of $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$.

An example

What can you tell about the following integral?

$$\int_1^{+\infty} \frac{1}{x^2 + x} dx$$

What's wrong with the following computation?

$$\begin{aligned}\int_{-1}^{1} \frac{1}{x} dx &= \left[\ln(|x|)\right]_{-1}^{1} \\ &= \ln(|1|) - \ln(|-1|) \\ &= 0\end{aligned}$$

A first step to the BCT – *Homework*

Let $f : [a, +\infty) \to \mathbb{R}$ a function such that:

- $\forall c \in [a, +\infty)$, f is integrable on $[a, c]$, and,
- $\forall x \in [a, +\infty)$, $f(x) \geq 0$.

Prove that $\int_a^{+\infty} f(t)dt$ is convergent if and only if

$$\left\{ \int_a^c f(t)dt,\ c \in [a, +\infty) \right\}$$

is bounded from above.