Theorems about sequences

February 13th, 2019

For after the reading week

For Monday (Feb 25), watch the videos:

- Improper integrals: 12.1, (12.2), (12.3), 12.4, (12.5), (12.6)

For Wednesday (Feb 27), watch the videos:

- BCT and LCT for integrals: 12.7, 12.8, 12.9, 12.10

Review – True or False

1. If a sequence is convergent, then it is bounded from above.
2. If a sequence is convergent, then it is eventually monotonic.
3. If a sequence diverges and is increasing, then there exists $n \in \mathbb{N}$ such that $a_n > 100$.
4. If $\lim_{n\to\infty} a_n = L$, then $a_n < L + 1$ for all n.
5. If a sequence is non-decreasing and non-increasing, then it is convergent.
6. If a sequence isn't decreasing and isn't increasing, then it is convergent.

True or False?

Let f be a function with domain at least $[0, \infty)$.
We define a sequence $(a_n)_{n\in\mathbb{N}}$ as $a_n = f(n)$.
Let $L \in \mathbb{R}$.

1. IF $\lim_{x\to\infty} f(x) = L$, THEN $\lim_{n\to\infty} a_n = L$.
2. IF $\lim_{n\to\infty} a_n = L$, THEN $\lim_{x\to\infty} f(x) = L$.
3. IF $\lim_{n\to\infty} a_n = L$, THEN $\lim_{n\to\infty} a_{n+1} = L$.

True or False – Monotonic sequences vs functions

Let f be a function with domain $[0, \infty)$.
We define the sequence $(a_n)_{n \geq 0}$ by $a_n = f(n)$.

1. IF f is increasing, THEN $(a_n)_n$ is increasing.

2. IF $(a_n)_n$ is increasing, THEN f is increasing.

Comment about sequences defined using a function

Beware: do not confuse sequences defined by

$$R_n = f(n)$$

(as in the two previous slides) and sequences defined by induction

$$R_{n+1} = f(R_n)$$

(as last Monday).

Compute

1. $$\lim_{n\to\infty} \frac{n! + 2e^n}{3n! + 4e^n}$$

2. $$\lim_{n\to\infty} \frac{2^n + (2n)^2}{2^{n+1} + n^2}$$

3. $$\lim_{n\to\infty} \frac{5n^5 + 5^n + 5n!}{n^n}$$

Big Theorem – True or False

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be positive sequences.

1. IF $a_n << b_n$, THEN $\forall m \in \mathbb{N},\ a_m < b_m$.
2. IF $a_n << b_n$, THEN $\exists m \in \mathbb{N}$ s.t. $a_m < b_m$.
3. IF $a_n << b_n$, THEN $\exists n_0 \in \mathbb{N}$ s.t. $\forall m \in \mathbb{N},\ m \geq n_0 \implies a_m < b_m$.
4. IF $\forall m \in \mathbb{N},\ a_m < b_m$, THEN $a_n << b_n$.
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1. Write a version of the Squeeze Theorem for convergent sequences.

2. Write a comparison theorem for sequences divergent to $+\infty$.

Homework: prove them!

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Almost increasing

A sequence $(a_n)_{n\in\mathbb{N}}$ is called *2-increasing* when

$$\forall n \in \mathbb{N}, \quad a_n < a_{n+2}.$$

Construct a sequence that is 2-increasing but not increasing.

Convergent sequence of integers – Homework

Prove that if a sequence of integers is convergent then it is eventually constant.

Composition law – Homework

Write a proof for the following Theorem

Theorem

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence. Let $L \in \mathbb{R}$.

- IF $\begin{cases} \lim\limits_{n\to+\infty} a_n = L \\ f \text{ is continuous at } L \end{cases}$
- THEN $\lim\limits_{n\to+\infty} f(a_n) = f(L)$.

Prove, directly from the definition of limit, that

$$\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1.$$