SEQUENCES: DEFINITION AND FIRST PROPERTIES

UNIVERSITY OF TORONTO

February 11$^{\text{th}}$, 2019

For next lecture

For Wednesday (Feb 13), watch the videos:

- Theorems about sequences: 11.5, 11.6, 11.7, 11.8

Warm up

Write a formula for the general term of these sequences

1. $(a_n)_{n\geq1} = (\,1,\ 4,\ 9,\ 16,\ 25,\ \dots\,)$
2. $(b_n)_{n\geq1} = (\,1,\ -2,\ 4,\ -8,\ 16,\ -32,\ \dots\,)$
3. $(c_n)_{n\geq0} = (\,1,\ -2,\ 4,\ -8,\ 16,\ -32,\ \dots\,)$
4. $(d_n)_{n\geq1} = \left(\frac{2}{1!},\ \frac{3}{2!},\ \frac{4}{3!},\ \frac{5}{4!},\ \dots\right)$
5. $(e_n)_{n\geq0} = (\,1,\ 4,\ 7,\ 10,\ 13,\ \dots\,)$

Let $(a_n)_{n\geq K}$ be a sequence. Write the formal definition of the following notions:

1. $\lim\limits_{n\to+\infty} a_n = L$
2. $(a_n)_n$ is convergent.
3. $(a_n)_n$ is divergent.
4. $\lim\limits_{n\to+\infty} a_n = +\infty$

Definition of limit of a sequence

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence. Let $L \in \mathbb{R}$.
Which of these statements are equivalent to $\lim_{n\to+\infty} a_n = L$?

1. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
2. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\quad n > n_0 \implies |L - a_n| < \varepsilon.$
3. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{R},\ \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
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5. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| \leq \varepsilon.$
6. $\forall \varepsilon \in (0,1),\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
7. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \frac{1}{\varepsilon}.$
8. $\forall \varepsilon \in (0,1),\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \frac{1}{\varepsilon}.$
9. $\forall k \in \mathbb{Z}_{>0},\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < k.$
10. $\forall k \in \mathbb{Z}_{>0},\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \frac{1}{k}.$

A sequence defined by recurrence

Consider the sequence $(R_n)_{n\geq 0}$ defined by

$$\begin{cases} & R_0 = 1 \\ \forall n \in N, & R_{n+1} = \dfrac{R_n + 2}{R_n + 3} \end{cases}$$

Compute $R_1,\ R_2,\ R_3$.

Is this proof correct?

Let $(R_n)_{n\geq 0}$ be the sequence in the previous slide.

Claim:

$$\lim_{n\to+\infty} R_n = -1+\sqrt{3}.$$

Proof.

- Let $L = \lim_{n\to\infty} R_n$.
- $R_{n+1} = \dfrac{R_n+2}{R_n+3}$
- $\lim_{n\to\infty} R_{n+1} = \lim_{n\to\infty} \dfrac{R_n+2}{R_n+3}$
- $L = \dfrac{L+2}{L+3}$
- $L(L+3) = L+2$
- $L^2+2L-2=0$
- $L=-1\pm\sqrt{3}$
- L must be positive, so $L=-1+\sqrt{3}$

□

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- $L = \frac{L+2}{L+3}$
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1. Prove that $(R_n)_{n\geq 0}$ is bounded from below by 0.
2. Prove that $(R_n)_{n\geq 0}$ is decreasing.
3. Prove that $(R_n)_{n\geq 0}$ is convergent.
4. Conclude.