University of Toronto – MAT137Y1 – LEC0501

Calculus!

Notes about slides 4 and 5

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Disclaimer: those are *quick-and-dirty* notes written just after the class, so it is very likely that they contain some mistakes/typos...
Send me an e-mail if you find something wrong/suspicious and I will update the notes.

The following criterion can be very useful to prove that a function is integrable!

Theorem 1 (from slide 4)**.**
Let f be a bounded function on $[a, b]$.
Then f is integrable on $[a, b]$ if and only if

$$\forall \varepsilon > 0,\ \exists \text{ a partition } P \text{ of } [a,b],\ U_P(f) - L_P(f) < \varepsilon$$

Proof.
$\Rightarrow$:
We know that f is integrable on $[a, b]$, i.e.

$$\underline{I_a^b}(f) = \overline{I_a^b}(f) \tag{1}$$

where

$$\underline{I_a^b}(f) = \sup\left\{L_P(f),\ \forall P \text{ partition of [a,b]}\right\} \qquad \text{and} \qquad \overline{I_a^b}(f) = \inf\left\{U_P(f),\ \forall P \text{ partition of [a,b]}\right\}$$

We want to prove:

$$\forall \varepsilon > 0,\ \exists \text{ a partition } P \text{ of } [a,b],\ U_P(f) - L_P(f) < \varepsilon$$

Let $\varepsilon > 0$.

Then $\overline{I_a^b}(f) + \frac{\varepsilon}{2}$ is greater than $\overline{I_a^b}(f)$ which is the greatest lower bound of the upper Darboux sums.
Hence $\overline{I_a^b}(f) + \frac{\varepsilon}{2}$ is not an lower bound of the upper Darboux sums.
That means that there exists a partition P_1 of $[a, b]$ such that

$$U_{P_1}(f) < \overline{I_a^b}(f) + \frac{\varepsilon}{2}$$

Similarly $\underline{I}_a^b(f) - \frac{\varepsilon}{2}$ is less than $\underline{I}_a^b(f)$ which is the least upper bound of the lower Darboux sums.
Hence $\underline{I}_a^b(f) - \frac{\varepsilon}{2}$ is not an upper bound of the lower Darboux sums.
That means that there exists a partition P_2 of $[a, b]$ such that

$$L_{P_2}(f) > \underline{I}_a^b(f) - \frac{\varepsilon}{2}$$

Let $P = P_1 \cup P_2$.
Then P is finer than P_1, hence

$$U_P(f) \leq U_{P_1}(f) < \overline{I}_a^b(f) + \frac{\varepsilon}{2} \tag{2}$$

and similarly P is finer than P_2, hence

$$L_P(f) \geq L_{P_2}(f) > \underline{I}_a^b(f) - \frac{\varepsilon}{2} \tag{3}$$

We derive from (2) and (3) that

$$U_P(f) - L_P(f) < \overline{I}_a^b(f) + \frac{\varepsilon}{2} - \underline{I}_a^b(f) + \frac{\varepsilon}{2}$$

Using (1), we obtain that the RHS of the above inequality is ε.
Therefore we have well obtained a partition P of $[a, b]$ such that

$$U_P(f) - L_P(f) < \varepsilon$$

Which is what we wanted to prove.

$\Leftarrow$:
We know that

$$\forall \varepsilon > 0, \exists \text{ a partition } P \text{ of } [a, b], U_P(f) - L_P(f) < \varepsilon$$

and we want to prove that f is integrable, i.e. that

$$\underline{I}_a^b(f) = \overline{I}_a^b(f)$$

It is enough to prove that

$$\forall \varepsilon > 0, 0 \leq \overline{I}_a^b(f) - \underline{I}_a^b(f) < \varepsilon$$

Let $\varepsilon > 0$.
By our assumption, there exists a partition P of $[a, b]$ such that $U_P(f) - L_P(f) < \varepsilon$.
Then, we have

$$L_P(f) \leq \underline{I}_a^b(f) \leq \overline{I}_a^b(f) \leq U_P(f)$$

Hence

$$0 \leq \overline{I}_a^b(f) - \underline{I}_a^b(f) \leq U_P(f) - L_P(f) < \varepsilon$$

We have well obtained

$$0 \leq \overline{I}_a^b(f) - \underline{I}_a^b(f) < \varepsilon$$

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The following proof is a good application of the above criterion.

Theorem 2 (From slide 5)**.** *If $f : [a,b] \to \mathbb{R}$ is non-decreasing then f is integrable on $[a,b]$.*

Remark 3. Notice that we don't assume that f is continuous, only that f is non-decreasing!

Proof. First, notice that f is bounded. Indeed, for any $x \in [a,b]$ we have $a \leq x \leq b$ and hence, since f is non-decreasing, we have

$$f(a) \leq f(x) \leq f(b)$$

Hence, f is bounded from above by $f(b)$ and from below by $f(a)$.

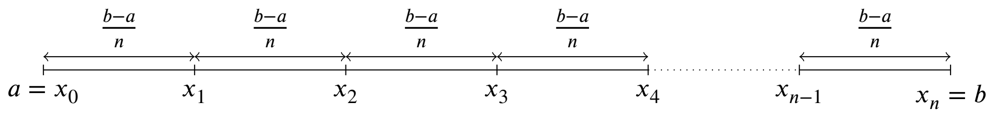
Then, according to the above criterion, it is enough to prove that

$$\forall \varepsilon > 0,\ \exists \text{ a partition } P \text{ of } [a,b],\ U_P(f) - L_P(f) < \varepsilon$$

Let $\varepsilon > 0$.
Set $n = \left\lfloor \frac{(f(b)-f(a))(b-a)}{\varepsilon} \right\rfloor + 1$. Then

$$\frac{(f(b)-f(a))(b-a)}{n} < \varepsilon \tag{4}$$

Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be the partition of $[a,b]$ consisting in n subintervals of the same length, i.e. $x_k = a + k\frac{b-a}{n}$.



Since f is non-decreasing, we easily check (*do it!*) that

$$\sup_{[x_{k-1},x_k]} f = f(x_k) \qquad \text{and} \qquad \inf_{[x_{k-1},x_k]} f = f(x_{k-1})$$

Then

$$U_P(f) = \sum_{k=1}^{n}\left((x_k - x_{k-1}) \sup_{[x_{k-1},x_k]} f\right) = \sum_{k=1}^{n}\left(\frac{b-a}{n} f(x_k)\right) = \frac{b-a}{n}\sum_{k=1}^{n} f(x_k)$$

and

$$L_P(f) = \sum_{k=1}^{n}\left((x_k - x_{k-1}) \inf_{[x_{k-1},x_k]} f\right) = \sum_{k=1}^{n}\left(\frac{b-a}{n} f(x_{k-1})\right) = \frac{b-a}{n}\sum_{k=1}^{n} f(x_{k-1})$$

Therefore

$$\begin{aligned}
U_P(f) - L_P(f) &= \frac{b-a}{n}\sum_{k=1}^{n}\left(f(x_k) - f(x_{k-1})\right) \\
&= \frac{b-a}{n}\left(f(x_1) - f(x_0) + f(x_2) - f(x_1) + f(x_3) - f(x_2) + \cdots + f(x_n) - f(x_{n-1})\right) \\
&= \frac{b-a}{n}(f(x_n) - f(x_0)) \\
&= \frac{b-a}{n}(f(b) - f(a))
\end{aligned}$$

We deduce from (4) that

$$U_P(f) - L_P(f) < \varepsilon$$

which is what we wanted to prove!

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