INTEGRABLE FUNCTIONS – 2

UNIVERSITY OF TORONTO

January 14th, 2019

For next lecture

For Wednesday (Jan 16), watch the videos:

- Riemann sums: 7.10, 7.11, 7.12
- Antiderivatives and indefinite integrals: 8.1, 8.2

Partitions of different intervals

1. Let P be a partition of $[0, 1]$.
   Let Q be a partition of $[1, 2]$.
   How do we construct a partition of $[0, 2]$ from them?

2. Let R be a partition of $[0, 2]$.
   How do we construct partitions of $[0, 1]$ and $[1, 2]$ from it?

The "ε–criterion" for integrability

Theorem

Let f be a bounded function on $[a, b]$.
Then f is integrable on $[a, b]$ if and only if

$$\forall \varepsilon > 0, \exists \text{ a partition } P \text{ of } [a, b], U_P(f) - L_P(f) < \varepsilon$$

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$\Rightarrow$:

1. Recall the definition of "f is integrable on $[a, b]$".
2. Fix $\varepsilon > 0$.
3. Show there exists a partition P_1 such that
$$U_{P_1}(f) < \overline{I_a^b}(f) + \frac{\varepsilon}{2}$$
4. Show there exists a partition P_2 such that
$$L_{P_2}(f) > \underline{I_a^b}(f) - \frac{\varepsilon}{2}$$
5. Using P_1 and P_2 from the previous step, construct a partition P such that $U_P(f) - L_P(f) < \varepsilon$.
6. Write the proof properly.

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$\Leftarrow$:

1. Fix $\varepsilon > 0$. Denote by P the partition given by the statement.
2. Order the quantities $U_P(f)$, $L_P(f)$, $\overline{I_a^b}(f)$, $\underline{I_a^b}(f)$.
3. Order the quantities $U_P(f) - L_P(f)$, $\overline{I_a^b}(f) - \underline{I_a^b}(f)$, 0, and ε.
4. Conclude.

Prove that a monotonic function is integrable.

Prove the following result.

Theorem

If $f : [a, b] \to \mathbb{R}$ is non-decreasing then f is integrable on $[a, b]$.

Preliminary question: Is f bounded?

Hint 1: Given a partition $P = \{x_0, x_1, \ldots, x_N\}$ of $[a, b]$, find simple expressions of the Darboux sums $U_P(f)$ and $L_P(f)$.

Hint 2: Do the same for the partition P of $[a, b]$ consisting in n subintervals of the same length. Then compute $U_P(f) - L_P(f)$.

Now, you have everything to write down a proof of the theorem!

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More on upper/lower sums[1]

1. Let f be a bounded function on $[0,1]$.
   Assume f is not constant.
   Prove that there exists a partition P of $[0,1]$ such that
   $$L_P(f) \neq U_P(f).$$

2. For which functions f is there a partition P of $[0,1]$ such that $L_P(f) = U_P(f)$?

[1] This slide was not covered in class. However, it is useful to practice Darboux sums.

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