MAT137Y1 – LEC0501

Calculus!

Integrable functions – 2

January 14th, 2019

For next lecture

For Wednesday (Jan 16), watch the videos:

- Riemann sums: 7.10, 7.11, 7.12
- Antiderivatives and indefinite integrals: 8.1, 8.2

Partitions of different intervals

1. Let P be a partition of $[0,1]$.
   Let Q be a partition of $[1,2]$.
   How do we construct a partition of $[0,2]$ from them?

2. Let R be a partition of $[0,2]$.
   How do we construct partitions of $[0,1]$ and $[1,2]$ from it?

The "ε–criterion" for integrability

Theorem

Let f be a bounded function on $[a,b]$.
Then f is integrable on $[a,b]$ if and only if

$$\forall \varepsilon > 0, \exists \text{ a partition } P \text{ of } [a,b],\ U_P(f) - L_P(f) < \varepsilon$$

Prove that a monotonic function is integrable.

Prove the following result.

Theorem

If $f : [a, b] \to \mathbb{R}$ is non-decreasing then f is integrable on $[a, b]$.

Preliminary question: Is f bounded?

Hint 1: Given a partition $P = \{x_0, x_1, \dots, x_N\}$ of $[a, b]$, find simple expressions of the Darboux sums $U_P(f)$ and $L_P(f)$.

Hint 2: Do the same for the partition P of $[a, b]$ consisting in n subintervals of the same length. Then compute $U_P(f) - L_P(f)$.

Now, you have everything to write down a proof of the theorem!

More on upper/lower sums[1]

1. Let f be a bounded function on $[0, 1]$.
   Assume f is not constant.
   Prove that there exists a partition P of $[0, 1]$ such that
   $$L_P(f) \neq U_P(f).$$

2. For which functions f is there a partition P of $[0, 1]$ such that $L_P(f) = U_P(f)$?

[1] This slide was not covered in class. However, it is useful to practice Darboux sums.