INTEGRABLE FUNCTIONS

January 9th, 2019

For next week

For Monday (Jan 14), watch the videos:

- Integrable functions: 7.8, 7.9

For Wednesday (Jan 16), watch the videos:

- Riemann sums: 7.10, 7.11, 7.12
- Antiderivatives and indefinite integrals: 8.1, 8.2

Warm up: suprema and infima

Find the supremum, infimum, maximum, and minimum of the following sets (if they exist):

1. $A = [-1, 5)$
2. $B = (-\infty, \pi)$
3. $C = \{\sqrt{2}, e, \pi\}$
4. $D = \mathbb{N}$
5. $E = \left\{\frac{1}{n} \ : \ n \in \mathbb{Z}, n > 0\right\}$
6. $F = \left\{\frac{(-1)^n}{n} \ : \ n \in \mathbb{Z}, n > 0\right\}$
7. $G = \{2^n \ : \ n \in \mathbb{Z}\}$
8. $H = (0, 1] \cap \mathbb{Q}$

Unicity of the supremum

Prove the following fact:

Theorem

If a subset $A \subseteq \mathbb{R}$ admits a supremum, then it is unique.

Recall:

Let $A \subseteq \mathbb{R}$. Let $S \in \mathbb{R}$.

- S is an **upper bound** of A means $\forall x \in A, x \leq S$.
- S is the **least upper bound** (or **supremum**) of A means
  - S is an upper bound of A, and,
  - for all upper bounds T of A, $S \leq T$.

The same result holds for the infimum.

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Empty set

1. Does $\varnothing$ have an upper bound ?
2. Does $\varnothing$ have a supremum?
3. Does $\varnothing$ have a maximum?
4. Is $\varnothing$ bounded from above?

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Equivalent definition of the supremum

For $A \subseteq \mathbb{R}$ and $S \in \mathbb{R}$, recall:

Definition

We say that S is the supremum of A, denoted by $S = \sup(A)$, if it is the least upper bound of A.

Write a formal definition using quantifiers.

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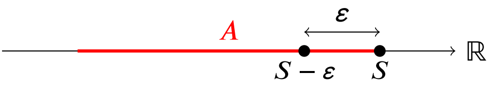

Beware: the above drawing is quite helpful, but keep in mind that there are subsets of $\mathbb{R}$ that are not intervals (there may be some “holes”)!

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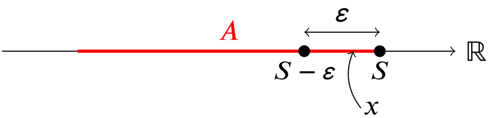

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Infima and suprema exercises

Let $A, B \subseteq \mathbb{R}$.
Which of the following statements are true or false?
If true, prove it. If false, find a counterexample.

1. If $B \subseteq A$ and A is bounded from above, then B is bounded from above.
2. If $B \subseteq A$ and B is bounded from above, then A is bounded from above.
3. If A and B each have a supremum and $B \subseteq A$, then $\sup B \leq \sup A$.
4. If A and B each have a supremum and $\sup B \leq \sup A$, then $B \subseteq A$.
5. If A and B each have a supremum, then $A \cup B$ has a supremum and $\sup(A \cup B) = \max\{\sup A, \sup B\}$.
6. If A and B each have a supremum then $A \cap B$ has a supremum and $\sup(A \cap B) = \min\{\sup A, \sup B\}$.
   What if moreover $A \cap B \neq \varnothing$?

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   What if moreover $A \cap B \neq \varnothing$? Fix the statement in this case.

False statements

Let $a \leq b$ and $f, g : [a, b] \to \mathbb{R}$ be two bounded functions.

For each of the following statements, find a counter-example and then fix the statement:

1. Let f and g be bounded functions on $[a, b]$. Then

$$\begin{matrix}\text{sup of } (f+g) \\ \text{on } [a,b]\end{matrix} \quad = \quad \begin{matrix}\text{sup of } f \\ \text{on } [a,b]\end{matrix} \quad + \quad \begin{matrix}\text{sup of } g \\ \text{on } [a,b]\end{matrix}$$

2. Let f be a bounded function on $[a, b]$. Let $c \in \mathbb{R}$. Then:

$$\begin{matrix}\text{sup of } (cf) \\ \text{on } [a,b]\end{matrix} \quad = \quad c \left(\begin{matrix}\text{sup of } f \\ \text{on } [a,b]\end{matrix} \right)$$

Warm up: lower and upper sums

Let $f(x) = \cos x$.

Consider the partition $P = \{0, 1, 2, 4\}$ of $[0, 4]$.

Compute the lower and upper Darboux sums of f with respect to P,
i.e. $L_P(f)$ and $U_P(f)$.