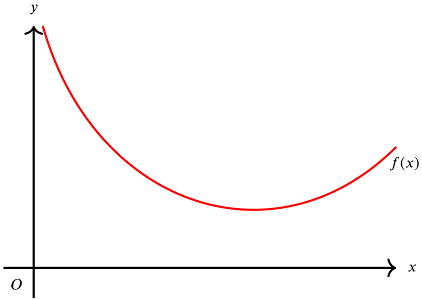
Concavity & Asymptotes

UNIVERSITY OF TORONTO

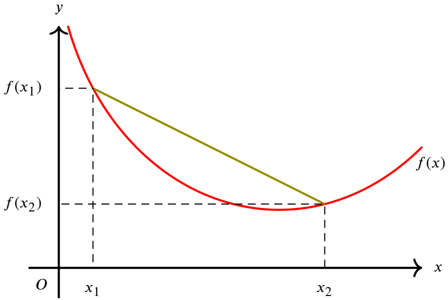
December 5th, 2018



y
f(x)
x
O

Definition: f is concave up on I

You've seen in the video that $f : I \to \mathbb{R}$ is concave up if the line segment between any two points on the graph of the function lies above or on the graph.



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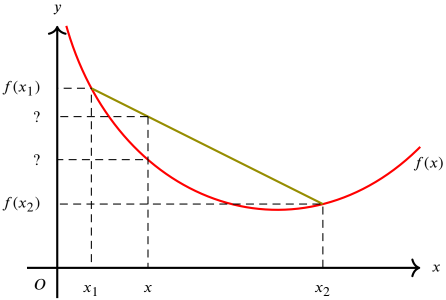

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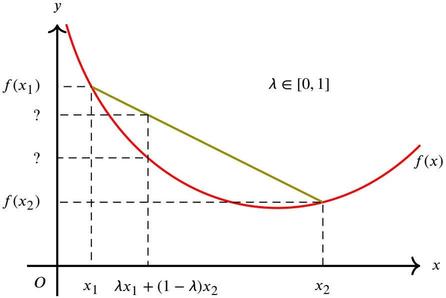

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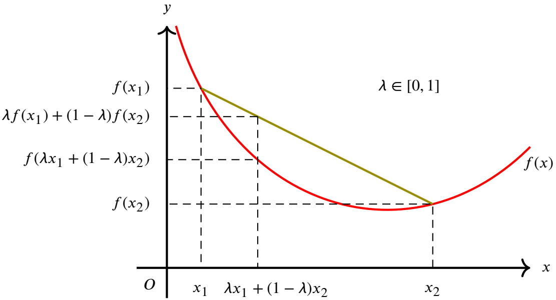

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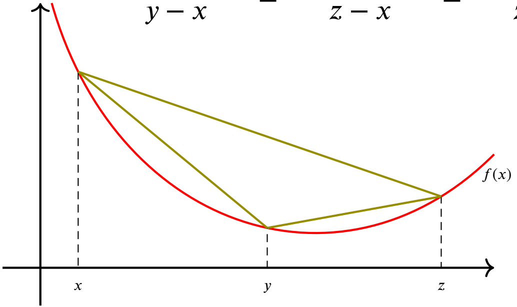
The three chords lemma

Let $f : I \to \mathbb{R}$ be a function defined on an interval I.

Prove that f is concave up on I if and only if $\forall x, y, z \in I$,

$$x < y < z \implies \frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y}$$



Hint: first rewrite each of these inequalities as $f(y) \leq \cdots$ and notice something that could simplify the question (and gives you a new definition for "f is concave up").

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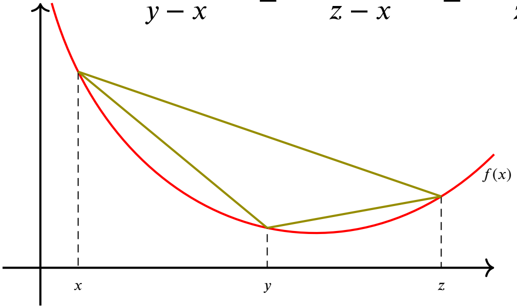
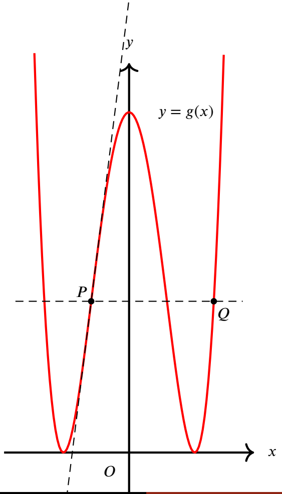


Corollary: *a concave up function $f : (a, b) \to \mathbb{R}$ is continuous.*
Warning: the openness of the interval is important, see f defined on $[0, 1)$ by $f(x) = 0$ on $(0, 1)$ and $f(0) = 1$.

$$g(x) = x^4 - 6x^2 + 9$$



Monotonicity and concavity

Let $f(x) = xe^{-x^2/2}$.

1. Find the intervals where f is increasing or decreasing, and its local extrema.
2. Find the intervals where f is concave up or concave down, and its inflection points.
3. Calculate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
4. Using this information, sketch the graph of f.

Unusual examples

Construct a function f such that

- the domain of f is at least $(0, \infty)$
- f is continuous and concave up on its domain
- $\lim_{x\to\infty} f(x) = -\infty$

Construct a function g such that

- the domain of g is $\mathbb{R}$
- g is continuous
- g has a local minimum at $x = 0$
- g has an inflection point at $x = 0$

Some inequalities *(not covered)*

1. **Jensen's inequality.**
   Let $f : I \to \mathbb{R}$ be a concave up function defined on an interval I. Let $x_1, \dots, x_n \in I$. Let $\lambda_1, \dots, \lambda_n \in [0, 1]$ such that $\lambda_1 + \dots + \lambda_n = 1$.
   Prove that
   $$f\left(\sum_{k=1}^{n} \lambda_k x_k\right) \leq \sum_{k=1}^{n} \lambda_k f(x_k)$$

2. Application: **AM–GM inequality.**
   Let $x_1, \dots, x_n \in (0, +\infty)$. Prove that
   $$\left(x_1 \cdots x_n\right)^{\frac{1}{n}} \leq \frac{x_1 + \dots + x_n}{n}$$

   Hint: Study $f(x) = -\ln(x)$.

A function with fractional exponents

Let $h(x) = \dfrac{x^{2/3}}{(x-1)^{2/3}}$.

This function is infinitely differentiable on $\mathbb{R} \setminus \{0, 1\}$ and

$$h'(x) = \frac{-2}{3x^{1/3}(x-1)^{5/3}} \qquad h''(x) = \frac{2(6x-1)}{9x^{4/3}(x-1)^{8/3}}$$

1. Find all asymptotes of h
2. Study the monotonicity of h and local extrema
3. Study the concavity of h and inflection points
4. With this information, sketch the graph of h