University of Toronto – MAT137Y1 – LEC0501

Calculus!

About Slide 7

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Disclaimer: those are *quick-and-dirty* notes written just after the class, so it is very likely that they contain some mistakes/typos...
Send me an e-mail if you find something wrong/strange and I will update the notes.

Question. *Prove that for every $x \geq 0$,*

$$\arcsin \frac{1-x}{1+x} + 2 \arctan \sqrt{x} = \frac{\pi}{2}$$

(*First, check that the above is well defined for $x \in [0, +\infty)$.*)

Let $f(x) = \arcsin \dfrac{1-x}{1+x} + 2 \arctan \sqrt{x}$.

First notice that $\frac{1-x}{1+x} = \frac{1+1-1-x}{1+x} = \frac{2}{1+x} - 1$. Then

$$\begin{aligned} & x \geq 0 \\ \Rightarrow \quad & 1 + x \geq 1 \\ \Rightarrow \quad & \frac{1}{1+x} \leq 1 \\ \Rightarrow \quad & \frac{2}{1+x} \leq 2 \\ \Rightarrow \quad & \frac{2}{1+x} - 1 \leq 1 \end{aligned}$$

Moreover, if $x \geq 0$ then $\frac{2}{1+x} - 1 > -1$.
So, for $x \geq 0$, $\frac{1-x}{1+x} = \frac{2}{1+x} - 1$ is in $(-1, 1]$ which is included in the domain of definition of arcsin ($\mathrm{dom}(\arcsin) = [-1, 1]$).
Besides, the domain of definition of $\sqrt{x}$ is $[0, +\infty)$ and the one of arctan is $\mathbb{R}$.
Hence f is defined and continuous on $[0, +\infty)$.

However, be careful that the domain of differentiability of $\sqrt{x}$ is $(0, +\infty)$ and that the domain of differentiability of arcsin is $(-1, 1)$.
By the differentiation rules, we only know that f is differentiable on $(0, +\infty)$. We can't say anything about the differentiability at 0 without providing additional work.

For $x \in (0, +\infty)$, we have

$$\begin{aligned} f'(x) &= -\frac{2}{(1+x)^2}\frac{1}{\sqrt{1-\left(\frac{1-x}{1+x}\right)^2}} + \frac{1}{\sqrt{x}}\frac{1}{1+\sqrt{x}^2} \\ &= -\frac{2}{1+x}\frac{1}{\sqrt{(1+x)^2-(1-x)^2}} + \frac{1}{\sqrt{x}}\frac{1}{1+x} \\ &= -\frac{2}{1+x}\frac{1}{\sqrt{4x}} + \frac{1}{\sqrt{x}}\frac{1}{1+x} \\ &= -\frac{1}{1+x}\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x}}\frac{1}{1+x} \\ &= 0 \end{aligned}$$

Since $f'(x) = 0$ on **the interval**[⋆] $(0, +\infty)$, we know that f is constant on **the interval** $(0, +\infty)$.
Therefore there exists $c \in \mathbb{R}$, such that $\forall x \in (0, +\infty),\ f(x) = c$.
Moreover, we know that f is continuous on $[0, +\infty)$, so $f(0) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} c = c$.
Hence f is constant on $[0, +\infty)$.
Let's compute f at any point, for instance 0:

$$c = f(0) = \arcsin(1) + 2\arctan(0) = \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

Conclusion: $\forall x \in [0, +\infty)$, $\arcsin \frac{1-x}{1+x} + 2\arctan\sqrt{x} = \frac{\pi}{2}$.

Remark. *Be careful, we only know that $f'(x) = 0$ on $(0, +\infty)$, not on $[0, +\infty)$.*
The continuity at 0 is crucial to prove that f is constant on $[0, +\infty)$!

You can use the following result (directly proved with the MVT).

Theorem. *Assume that f is continuous on an **interval I**, differentiable on the* interior *of I (i.e. omitting the endpoints if there are any).*
If for all x in the interior of I, $f'(x) = 0$ then f is constant on I.

Example. *We know that*
- *f is continuous on $[0, +\infty)$,*
- *f is differentiable on $(0, +\infty)$, and*
- *$\forall x \in (0, +\infty),\ f'(x) = 0$.*

Hence f is constant on $[0, +\infty)$.

⋆ As seen in Slide 8, this assumption is very important!