Rolle's Theorem & the MVT

UNIVERSITY OF TORONTO

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For next week

For Monday (Nov 26), watch the videos:

- Monotonicity: 5.10, 5.11, 5.12

For Wednesday (Nov 28), watch the videos:

- Applied optimization: 6.1, 6.2
- Indeterminate forms and L'Hôpital's Rule: 6.3, 6.4, 6.5, 6.6, 6.7

How many zeroes?

Let

$$f(x) = e^x - \sin x + x^2 + 10x$$

How many zeroes does f have?

A new theorem

We want to prove this theorem:

Theorem

Let f be a differentiable function defined on an interval I.
IF $\forall x \in I, f'(x) \neq 0$
THEN f is one-to-one on I.

Hint: work with the contrapositive!

A first theorem on monotonicity

Prove the following result:

Theorem

Let $a < b$. Let f be a function differentiable on (a, b).

- IF $\forall x \in (a, b), f'(x) > 0$,
- THEN f is increasing on (a, b).

1. Write formally what we want to show.
2. Write the structure of the proof.
3. Rough work!
4. Write a correct proof!

What is wrong with this proof?

Cauchy's MVT

Let $a < b$. Let f and g be functions defined on $[a, b]$.
IF

- f and g are continuous on $[a.b]$,
- f and g are differentiable on (a, b),
- $g(b) \neq g(a)$,
- $\forall x \in (a, b),\ g'(x) \neq 0$.

THEN $\exists c \in (a, b)$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

(Bad) Proof:

- By MVT, $\exists c \in (a, b)$ s.t. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$
- By MVT, $\exists c \in (a, b)$ s.t. $g'(c) = \dfrac{g(b) - g(a)}{b - a}$
- Divide the two equations and we get what we wanted.

What is wrong with this proof?

Cauchy's MVT

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Write a correct proof!

Hint : apply Rolle's theorem on $[a, b]$ to the function $H(x) = f(x) - Mg(x)$ for a suitable $M \in \mathbb{R}$.
Be careful: the M given during the rough work in class was correct, but then I made a typo on the blackboard!

Proving difficult identities

1. Prove that, for every $x \geq 0$,
$$\arcsin \frac{1-x}{1+x} + 2\arctan \sqrt{x} = \frac{\pi}{2}$$
(First, why is this function well defined on $[0, +\infty)$?)

2. Prove that, for every $x \in [-1, 1]$,
$$\arccos(x) + \arcsin(x) = \frac{\pi}{2}$$

3. Prove that, for every $x \in \mathbb{R}$,
$$\arctan(x) + 2\arctan\left(\sqrt{1+x^2} - x\right) = \frac{\pi}{2}$$

Proving difficult identities

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Hint: use differentiation!
But be careful about the domain of differentiability!

Is the following claim/proof correct?

Claim

$\forall x \in \mathbb{R} \setminus \{0\}, \arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$

Proof.

Let $f(x) = \arctan(x) + \arctan\left(\frac{1}{x}\right)$.

For $x \in \mathbb{R} \setminus \{0\}$, we have

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{x^2}\frac{1}{1+\frac{1}{x^2}} = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

Hence $f(x)$ is constant.

We conclude by noticing that $f(1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$. □

Prove the corrected claim

Theorem

$$\forall x \in \mathbb{R} \setminus \{0\},\ \arctan(x) + \arctan\left(\frac{1}{x}\right) = \operatorname{sgn}(x)\frac{\pi}{2}$$