University of Toronto – MAT137Y1 – LEC0501

Calculus!

About Slide 4

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Disclaimer: those are *quick-and-dirty* notes written just after the class, so it is very likely that they contain some mistakes/typos...
Send me an e-mail if you find something wrong/strange and I will update the notes.

Theorem (Local EVT *).
Let I be an interval and let $f : I \to \mathbb{R}$ be a function defined on I. Let $c \in I$.
Assume that c isn't an endpoint of I and that f admits a local extremum at c.
Then either $f'(c)$ doesn't exist or $f'(c) = 0$.

Remark. *Be careful with the assumptions of the above result: c has to be an interior point of I. Therefore, this result does not say anything on endpoints. When looking for global extrema, you have to treat the endpoints separately (maybe they are global extrema, maybe not).*

Remark. *Be careful with the conclusion: there is a disjunction, do not forget the case "$f'(c)$ doesn't exist". Study the following function around 0 to convince yourself:*

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

Remark. *The converse of the above theorem is false.*
Indeed, for instance, let $f(x) = x^3$. Then $f'(0) = 0$. But $f(0) = 0$ isn't a local extremum since $f(x) < 0$ for any negative x and $f(x) > 0$ for any positive x.
Study the following functions around 0 (they are a little bit trickier, so understanding why they disprove the converse is a good exercise):

$$g(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$h(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Remark. *When you are looking for the (global) extrema of a function $f : I \to \mathbb{R}$, if they exist, they are among:*
- *The endpoints of I.*
- *The interior points of I where f isn't differentiable.*
- *The interior points of I where the derivative of f vanishes.*

But keep in mind that (1) it is possible for a function to not have extrema (2) it is possible that the above points are not even local extrema as in the previous remark!
Hence it is not enough to give the list of such points. You need to provide additional work!

* This theorem is usually called *Fermat's theorem* (*for stationary points*).

The question on Slide 4 was:

Question. *Does the function* $f(x) = \dfrac{\sin(x)}{\sqrt{2} + \cos(x)}$ *has a min? a max? If so, explicit them!*

Step 1: Notice that f is 2π-periodic: $\forall x \in \mathbb{R},\ f(x + 2\pi) = f(x)$.
So it is enough to focus on an interval of length 2π, for instance $[0, 2\pi]$.
This step is important since it'll allow us to use the EVT in the next step. Remember that the interval must be a line segment to apply the EVT!

Step 2: By the Main Continuity Theorem, we know that f is continuous on $[0, 2\pi]$ which is a line segment. Hence, according to the EVT, f has both a min and a max on $[0, 2\pi]$.
This step ensures that f admits a min and a max! Now, we know they exist! That'll be very important for the next step.

Step 3: f is differentiable ewerywhere and $f'(x) = \dfrac{\cos(x)\left(\sqrt{2} + \cos(x)\right) + \sin^2(x)}{\left(\sqrt{2} + \cos(x)\right)^2} = \dfrac{\sqrt{2}\cos(x) + 1}{\left(\sqrt{2} + \cos(x)\right)^2}$.
So $f'(x) = 0 \Leftrightarrow \cos(x) = -\frac{1}{\sqrt{2}} \Leftrightarrow x = \frac{3\pi}{4}$ or $x = \frac{5\pi}{4}$ on $[0, 2\pi]$.
According to the local EVT, the min and the max are among the endpoints and the two above points: $0, \frac{3\pi}{4}, \frac{5\pi}{4}, 2\pi$.

- $f(0) = 0$.
- $f\left(\frac{3\pi}{4}\right) = 1$
- $f\left(\frac{5\pi}{4}\right) = -1$
- $f(2\pi) = 0$

We know from Step 2 (using the EVT) that f admits a max and a min on $[0, 2\pi]$ and from the local EVT that they are among the above points.
Hence the max of f on $[0, 2\pi]$ is 1 at $\frac{3\pi}{4}$.
And the min of f on $[0, 2\pi]$ is -1 at $\frac{5\pi}{4}$.
Notice that Step 2 is crucial here: we used that we know that f admits a min and a max! Otherwise it could have been possible that f doesn't have a min or a max and that one of the two above points has a horizontal tangent lines but without being a local extremum like with $f(x) = x^3$ at 0 in a remark above!

For the conclusion, don't forget that we reduced the study to $[0, 2\pi]$ using the 2π-periodicity of f! So the min and the max are reached at additional points!

Conclusion:

- f admits a max which is 1, reached at $x = \frac{3\pi}{4} + 2k\pi$ for $k \in \mathbb{Z}$.
- f has a min which is -1, reached at $x = \frac{5\pi}{4} + 2k\pi$ for $k \in \mathbb{Z}$.

Remark. *You will have very soon additional tools allowing you to obtain simpler solutions for this kind of questions when f is nice enough (i.e. differentiable everywhere).*