ONE-TO-ONE FUNCTIONS & INVERSE TRIG FUNCTIONS

UNIVERSITY OF TORONTO

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For next week

For Monday (Nov 19), watch the videos:

- Local extrema: 5.1, 5.2, 5.3, 5.4

For Wednesday (Nov 21), watch the videos:

- Rolle's Theorem: 5.5, 5.6
- The MVT: 5.7, 5.8, 5.9

One-to-one functions

1. Write the formal definition of
"$f : D \to \mathbb{R}$ is one-to-one".

2. Let f be the function defined on $\mathbb{R}$ by
$f(x) = 2x^3 + 7$.
Prove that f is one-to-one.

3. Let g be the function defined on $\mathbb{R}$ by
$g(x) = 2x^2 + 7$.
Prove that g is not one-to-one.

Strictly increasing functions are one-to-one

Let $f : D \to \mathbb{R}$ a function whose domain D is a subset of $\mathbb{R}$.

1. Recall the definition of "f is strictly increasing".

2. Recall the definition of "f is one-to-one" (from the previous slide).

3. Prove the following result.

Theorem

If f is strictly increasing then f is one-to-one.

4. Now let $D = \mathbb{R} \setminus \left\{ \frac{\pi}{2} + n\pi,\ n \in \mathbb{R} \right\}$.
   Is $\tan : D \to \mathbb{R}$ strictly increasing? Is it one-to-one?

5. What about $\tan_{|\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$, the restriction of $\tan$ to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$?

Composition of one-to-one functions

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$. Prove the following theorem.

Theorem

Let f and g be functions.
IF f and g are one-to-one,
THEN $f \circ g$ is one-to-one.

Suggestion:

1. Write the definition of what you want to prove.
2. Figure out the formal structure of the proof.
3. Complete the proof (use the hypotheses!)

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$.

Is the following claim TRUE or FALSE?
Prove it or give a counterexample.

Claim

Let f and g be functions.
IF $f \circ g$ is one-to-one,
THEN f is one-to-one.

Composition of one-to-one functions – 3

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$.

Is the following claim TRUE or FALSE?
Prove it or give a counterexample.

Claim

Let f and g be functions.
IF $f \circ g$ is one-to-one,
THEN g is one-to-one.

Composition and inverses

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$.

Let f and g be functions. Assume they each have an inverse.

Is $(f \circ g)^{-1} = f^{-1} \circ g^{-1}$?

- If YES, prove it.
- If NO, fix the statement.

If you do not know how to start, experiment with the functions

$$f(x) = x + 1, \qquad g(x) = 2x.$$

Definition of $\arctan$

1. Sketch the graph of $\tan$.
2. Prove that $\tan$ is not one-to-one.
3. Select the largest interval containing 0 such that the restriction of $\tan$ to it is one-to-one.
   Briefly explain why $\tan$ is one-to-one on this interval.
4. What's the range of $\tan$ restricted to the above interval?
5. We define $\arctan$ as the inverse of $\tan$ restricted to the above interval.
   What are the domain and the range of $\arctan$?
6. Sketch the graph of $\arctan$.

Definition of $\arctan$ – 2

Remember from the previous slide that $\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is defined as the inverse of $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$.

1. Fill: $\forall x \in \ldots, \forall y \in \ldots, \left(y = \tan(x) \Leftrightarrow x = \arctan(y)\right)$.
2. What can you say about $\tan(\arctan(x))$?
   And about $\arctan(\tan(x))$?
   (For which x are these functions defined? What are they equal to? Sketch their graphs.)
3. Compute:
   1. $\tan(\arctan(0))$
   2. $\tan\left(\arctan\left(\sqrt{2} + \pi\right)\right)$
   3. $\arctan(\tan(1))$
   4. $\arctan(\tan(3))$
   5. $\arctan\left(\tan\left(\frac{\pi}{2}\right)\right)$
   6. $\arctan(\tan(-6))$

Derivative of $\arctan$

Obtain (and prove) a formula for the derivative of $\arctan$.

Hint: Differentiate the identity

$$\forall t \in \ldots \qquad \tan(\arctan(t)) = t$$

Compute the derivative of

$$f(x) = 2x^2 \arctan(x^2) - \ln(x^4 + 1)$$

and simplify it as much as possible.