Continuity

October 10th, 2018

For next week

For Monday (Oct 15), watch the videos:

- Some limit computations: 2.19, 2.20

For Wednesday (Oct 17), watch the videos:

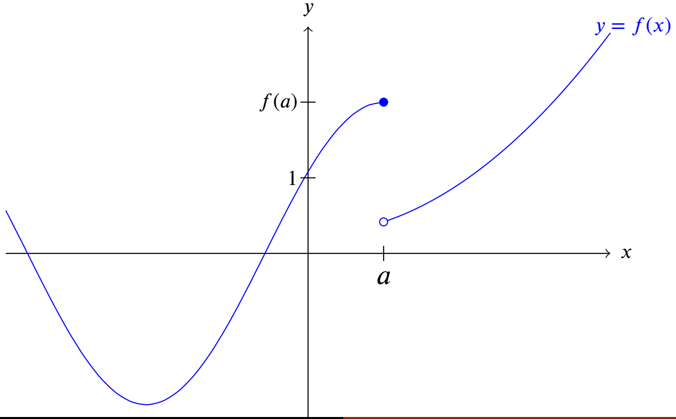
- IVT and EVT: 2.21, 2.22
- Definition of a derivative: 3.1, 3.2, 3.3

Graphical interpretation

Remember that a function f is continuous at a if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in \operatorname{dom}(f),\ \Big(|x-a| < \delta \implies |f(x)-f(a)| < \varepsilon \Big)$$

Write the negation.
Is the following function continuous at a? Explain why.



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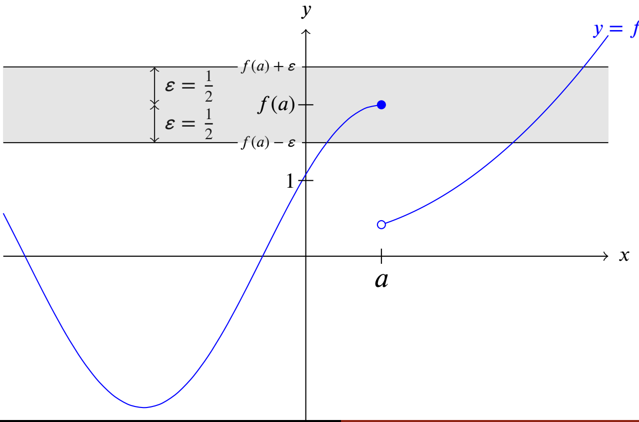

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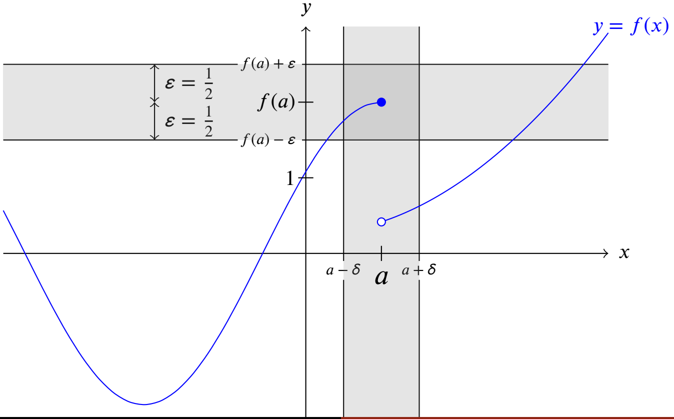

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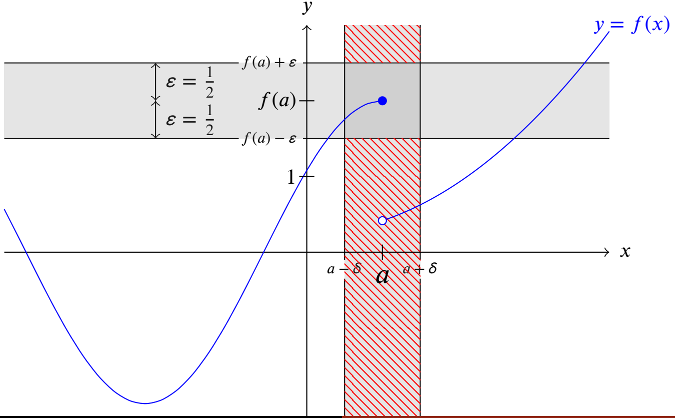

Undefined function

Let $a \in \mathbb{R}$ and
let f be a function defined on $\mathbb{R} \setminus \{a\} = (-\infty, a) \cup (a, +\infty)$.

What can we conclude?

1. $\lim_{x \to a} f(x)$ exist
2. $\lim_{x \to a} f(x)$ doesn't exist.
3. No conclusion. $\lim_{x \to a} f(x)$ may or may not exist.

What else can we conclude?

4. f is continuous at a.
5. f is not continuous at a.
6. No conclusion. f may or may not be continuous at a.

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New continuous functions

We want to prove the following theorem

Theorem

Let f and g be two functions.
IF f and g are continuous
THEN $h(x) = \max(f(x), g(x))$ is also a continuous function.

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What is the fastest way to prove this?

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Hint: What is the number $\frac{a+b+|a-b|}{2}$?
There is a way to prove this quickly without writing any epsilons.

Is this function continuous?

Consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x < 1 \\ x^2 & \text{if } 1 \leq x \leq 2 \\ \sqrt{8x} & \text{if } x > 2 \end{cases}$$

1. Sketch its graph.
2. Conclude graphically.
3. Conclude formally.

Is this function continuous?

Write a "two-line proof" of:

Claim

$$\lim_{x\to a} f(x) = 0 \Leftrightarrow \lim_{x\to a} |f(x)| = 0$$

Consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin(x)\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Is f continuous at 0?

We will see another method on slide 12.

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True or False?

Assuming these limits exist

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

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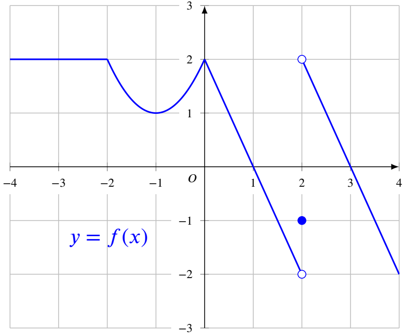
Remember:



What are $\lim_{x\to 0} f(f(x))$ and $f\left(\lim_{x\to 0} f(x)\right)$?

Fix

What extra condition can we add on f for this to be true?

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

A composition theorem

Write a formal proof for

Theorem

Let $a, L \in \mathbb{R}$.
Let f and g be functions with domain $\mathbb{R}$.
IF

- $\lim_{x \to a} g(x) = L$, and
- f is continuous at L

THEN $\lim_{x \to a} f(g(x)) = f(L)$.

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Under these assumptions:

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

True or False?

Is this theorem true?

Claim

Let $a \in \mathbb{R}$.
Let f and g be functions defined near a.

- IF $\lim_{x \to a} f(x) = 0$,
- THEN $\lim_{x \to a} [f(x)g(x)] = 0$.

A new theorem about products and limits

Theorem

Let $a \in \mathbb{R}$. Let f and g be functions with domain $\mathbb{R}$, except possibly a.

IF

- $\lim_{x \to a} f(x) = 0$, and
- g is bounded, i.e. $\exists M > 0, \forall x \neq a, |g(x)| \leq M$.

THEN $\lim_{x \to a} \left(f(x)g(x) \right) = 0$.

1. Write down formally what you want to prove.
2. Write down the structure of the formal proof.
3. Rough work.
4. Write down a complete formal proof.

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Application: prove that $\lim_{x \to 0} \sin(x) \sin\left(\frac{1}{x}\right) = 0$.

Change of Variables

On Oct 3, slide 5, question 3, we used a light version of:

Theorem

Let f be a function defined around a (but maybe not at a) and g be a function defined around b (but maybe not at b).

IF

1. $\lim_{t \to b} g(t) = a$,
2. $\lim_{x \to a} f(x) = L$,
3. $g(t) \neq a$ around b (but not at b), i.e.
   $\exists \eta > 0, \forall t \in \operatorname{dom}(g), 0 < |t - b| < \eta \implies g(t) \neq a$.

THEN $\lim_{t \to b} f(g(t)) = L$

⚠ Warning: there are **THREE** assumptions here.

If f is continuous, better use the result slide 10.

Composition and limits

Construct two functions f and g such that

$$\lim_{x \to 1} f(x) = 2$$

and

$$\lim_{x \to 0} g(x) = 1$$

and

$$\lim_{x \to 0} f(g(x)) = 42$$

Hint: use slides 10 and 13 to find conditions on f and g.