University of Toronto – MAT137Y1 – LEC0501

Calculus!

Slide 8: proof of non-existence

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Disclaimer: those are *quick-and-dirty* notes written just after the class, so there are probably some small mistakes/typos...
Send me an e-mail if you find something wrong/strange and I will update the notes.

Question: Prove that $\lim_{x\to 0^+} \frac{1}{x}$ doesn't exist.

Remember that $\lim_{x\to 0^+} \frac{1}{x}$ exists if there exists $L \in \mathbb{R}$ such that $\lim_{x\to 0^+} \frac{1}{x} = L$, i.e.

$$\exists L \in \mathbb{R},\ \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in \mathbb{R} \setminus \{0\},\ \left(0 < x < \delta \implies \left|\frac{1}{x} - L\right| < \varepsilon\right)$$

First I take $x \in \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, +\infty)$, because it is the domain of $f(x) = \frac{1}{x}$.
Then I focus only on x satisfying $0 < x < \delta$ because it is a one-sided limit at 0^+: we are interested in the behaviour of $f(x) = \frac{1}{x}$ when x is close to 0, but greater than 0.
Remember that:

- for a limit at a (both-sided, usual limit), we assume $0 < |x - a| < \delta$ (because we want to work with x close to a but not a),
- for a limit at a^+ (right-sided), we assume $0 < x - a < \delta$ ($\Leftrightarrow a < x < a + \delta$, because we want to work with x close to a but greater than a),
- for a limit at a^- (left-sided), we assume $0 < a - x < \delta$ ($\Leftrightarrow a - \delta < x < a$, because we want to work with x close to a but less than a),

What we want to show: the negation of the above statement, i.e.

$$\forall L \in \mathbb{R},\ \exists \varepsilon > 0,\ \forall \delta > 0,\ \exists x \in \mathbb{R} \setminus \{0\},\ \left(0 < x < \delta \text{ and } \left|\frac{1}{x} - L\right| \geq \varepsilon\right)$$

The difficult part consists in finding suitable ε and x.

Rough work:

$$\begin{aligned}\left|\frac{1}{x} - L\right| &\geq \left|\left|\frac{1}{x}\right| - |L|\right| && \text{by the reverse triangle inequality}\\ &\geq \left|\frac{1}{x}\right| - |L| \\ &= \frac{1}{x} - |L| && \text{since } x > 0\end{aligned}$$

So, it is enough to find ε and x such that

$$\begin{aligned} &\frac{1}{x} - |L| \geq \varepsilon \\ \Leftrightarrow\ &\frac{1}{x} \geq \varepsilon + |L| \\ \Leftrightarrow\ &x \leq \frac{1}{\varepsilon + |L|} \end{aligned}$$

There is no constraint on ε, so we can take $\varepsilon = 1$.
Then we need to have $x \leq \frac{1}{1+|L|}$.
Remember that we also want $0 < x < \delta$.
So we can take $x = \min\left(\frac{1}{1+|L|}, \frac{\delta}{2}\right)$.

The rough work is done... Now you can start to write the proof.

Proof:
Let $L \in \mathbb{R}$.
Fix $\varepsilon = 1$, we have $\varepsilon > 0$.
Let $\delta > 0$.
Fix $x = \min\left(\frac{1}{1+|L|}, \frac{\delta}{2}\right)$, we have $x \neq 0$.
Then $0 < x$ since either $x = \frac{1}{1+|L|} > 0$ or $x = \frac{\delta}{2} > 0$, and $x \leq \frac{\delta}{2} < \delta$.
Furthermore

$$\begin{aligned} \left|\frac{1}{x} - L\right| &\geq \left|\left|\frac{1}{x}\right| - |L|\right| \quad \text{by the reverse triangle inequality} \\ &\geq \left|\frac{1}{x}\right| - |L| \\ &= \frac{1}{x} - |L| \quad \text{since } x > 0 \\ &\geq 1 + |L| - |L| \quad \text{since } x \leq \frac{1}{1+|L|} \\ &= 1 = \varepsilon \end{aligned}$$

Hence we have well

$$0 < x < \delta$$

and

$$|f(x) - L| \geq \varepsilon$$

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