University of Toronto – MAT137Y1 – LEC0501

Calculus!

Slide 6: your first $\varepsilon - \delta$ proof

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Disclaimer: those are *quick-and-dirty* notes written just after the class, so there are probably some small mistakes/typos...
Send me an e-mail if you find something wrong/strange and I will update the notes.

Question: Prove that $\lim_{x\to 1}(x^2+x)=2$.

What we want to show: $\forall\varepsilon > 0, \exists\delta > 0, \forall x \in \mathbb{R}, \left(0 < |x-1| < \delta \implies |x^2+x-2| < \varepsilon\right)$

The difficult part consists in finding a suitable δ. Notice that δ is introduced before x in the above statement, so δ can't be defined in terms of x, only in terms of ε.

Rough work: There is no trick, formula or algorithm to find a suitable δ. At first glance, such proofs will look like difficult/abstract. However, the more you will practice, the more comfortable you will be with this kind of proofs. Remember, *practice makes perfect!*
The "rough work" you used to find δ is not part of the proof. It should stay in your draft (you shouldn't submit it). We don't care about how you found δ.
In the proof, you'll propose a δ, and prove that it works. You only have to explain why it works (that's the proof).

Here is my strategy for the rough work of this exercise: first we are going to assume that we have a $\delta > 0$ such that $0 < |x-1| < \delta$ and then we will manipulate the conclusion we would like to have ($|x^2+x-2| < \varepsilon$) in order to find conditions on δ for the conclusion to be true.
Remember, the rough work stays in your draft: it doesn't need to be 100% mathematically correct. Just enough to give you hints to find a good δ (then you will have to write a 100% mathematically/logically/syntactically correct proof).

Since we only know that $0 < |x-1| < \delta$, we need to perform algebraic manipulations on $|x^2+x-2|$ to make $|x-1|$ appears (and no other occurence of x: remember, δ can't depend on x).

$$
\begin{aligned}
\left|x^2+x-2\right| &= |(x-1)(x+2)| \\
&= |x-1|\cdot|x+2| && \text{Good, } |x-1| \text{ appears! But there is still another } x \text{ that we should remove.} \\
&= |x-1|\cdot|(x-1)+3| && \text{Just write } x=(x-1)+1! \\
&\le |x-1|\left(|x-1|+3\right) && \text{By the triangle inequality.} \\
&= |x-1|^2+3|x-1| \\
&< \delta^2+3\delta && \text{Because } 0<|x-1|<\delta
\end{aligned}
$$

If we find a δ such that $\delta^2 + 3\delta \leq \varepsilon$, we win! Notice that it is OK to work with a large $\leq$ here (and not a strict $<$). Indeed we already obtained a strict $<$ in the last step.
Notice it is enough to find a δ such that $\delta^2 \leq \frac{\varepsilon}{2}$ and $3\delta \leq \frac{\varepsilon}{2}$. Indeed, then

$$\delta^2 + 3\delta \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

But

$$\delta^2 \leq \frac{\varepsilon}{2} \Leftrightarrow \delta \leq \sqrt{\frac{\varepsilon}{2}} \quad \text{since the involved numbers are positive}$$

and

$$3\delta \leq \frac{\varepsilon}{2} \Leftrightarrow \delta \leq \frac{\varepsilon}{6}$$

So if we take $\delta = \min\left(\sqrt{\frac{\varepsilon}{2}}, \frac{\varepsilon}{6}\right)$, both inequalities would be satisfied and we would obtain

$$\left|x^2 + x - 2\right| < \delta^2 + 3\delta \leq \varepsilon$$

as wanted!

The rough work is done... Now you can start to write the proof.

Proof:
Let $\varepsilon > 0$.
Fix $\delta = \min\left(\sqrt{\frac{\varepsilon}{2}}, \frac{\varepsilon}{6}\right)$.
Let $x \in \mathbb{R}$.
Assume that $0 < |x - 1| < \delta$. Then

$$\begin{aligned}
\left|x^2 + x - 2\right| &= |(x-1)(x+2)| \\
&= |x-1| \cdot |x+2| \\
&= |x-1| \cdot |(x-1)+3| \\
&\leq |x-1|\big(|x-1|+3\big) \\
&= |x-1|^2 + 3|x-1| \\
&< \delta^2 + 3\delta \quad \text{Since } |x-1| < \delta \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{Since } 0 < \delta \leq \sqrt{\frac{\varepsilon}{2}} \text{ and } \delta \leq \frac{\varepsilon}{6} \\
&= \varepsilon
\end{aligned}$$

So $\left|x^2 + x - 2\right| < \varepsilon$.

∎

There are many possibilities for δ, you just need to find one, and to explain why it works...
Using the method described in the video, you would obtain $\delta = \min\left(1, \frac{\varepsilon}{4}\right)$ which also works.
Indeed, then $|x-1| < \delta \leq 1 \implies |x+2| < 4$. And $|x-1| < \delta \leq \frac{\varepsilon}{4}$.
Thus

$$\left|x^2 + x - 2\right| = |x-1| \cdot |x+2| < \frac{\varepsilon}{4}4 = \varepsilon$$