Definitions, proofs and induction

September 19th, 2018

For next week

For Monday (Sep 24), watch the videos:

- Absolute value, distance and inequalities: 2.4

For Wednesday (Sep 26), watch the videos:

- Limits: 2.1, 2.2, 2.3, 2.5, 2.6

Definition of an odd number

Pick the correct definition.

Definition

Let $n \in \mathbb{Z}$. We say that n is odd if

1. $n = 2a + 1$
2. $\forall a \in \mathbb{Z},\ n = 2a + 1$
3. $\exists a \in \mathbb{Z},\ n = 2a + 1$

True or false

Is the following statement true or false? Prove it!

Let $n, m \in \mathbb{Z}$. If none of n and m are multiple of 5, then $n + m$ is not multiple of 5.

A bad proof

What's wrong with the following proof?

Theorem

The sum of two odd numbers is even.

Proof.

3 is odd.
5 is odd.
$3+5=8$ is even. □

Write a correct proof!

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$\forall\exists$ and $\exists\forall$

Are the following statements true or false? Prove it!

1. $\exists n \in \mathbb{N}, \forall p \in \mathbb{N}, p \leq n$
2. $\forall p \in \mathbb{N}, \exists n \in \mathbb{N}, p \leq n$

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∀∃ and ∃∀

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⚠ You can **not** permute quantifiers of different kinds.

Prove the following statements

Statement 1

$$\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ x + y > 0$$

Statement 2

$$\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ y^2 > x$$

Prove the following statements

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Definition: one-to-one functions

A *one-to-one* (or injective) function is a function which maps two different elements to two different values.

Let $f : D \to \mathbb{R}$ be a function.
Which of the following statements mean that f is one-to-one?

1. $f(x_1) \neq f(x_2)$
2. $\exists x_1, x_2 \in D,\ f(x_1) \neq f(x_2)$
3. $\forall x_1, x_2 \in D,\ f(x_1) \neq f(x_2)$
4. $\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
5. $\forall x_1, x_2 \in D,\ x_1 = x_2 \implies f(x_1) = f(x_2)$
6. $\forall x_1, x_2 \in D,\ f(x_1) \neq f(x_2) \implies x_1 \neq x_2$
7. $\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$

Use mathematical symbols to write that f is not injective.

Is f one-to-one?

Definition

A function $f : D \to \mathbb{R}$ is said to be one-to-one when

$$\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$$

or equivalently when

$$\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

Prove that:

1. $f(x) = 3x + 5$ is one-to-one.
2. $f(x) = x^2$ is not one-to-one.

A bad proof using induction

Theorem

For $n \in \mathbb{N}$, 7 divides $9^n - 2^{n+1}$.

Proof:
We assume that the property is true for some n and we show that it also holds for $n+1$.
By our induction hypothesis, $\exists a \in \mathbb{N}$ such that $9^n - 2^{n+1} = 7a$.
Then,

$$\begin{aligned} 9^{n+1} - 2^{n+2} &= 9 \times 9^n - 2 \times 2^{n+1} \\ &= 9 \times (9^n - 2^{n+1} + 2^{n+1}) - 2 \times 2^{n+1} \\ &= 9 \times (9^n - 2^{n+1}) + 9 \times 2^{n+1} - 2 \times 2^{n+1} \\ &= 9 \times 7 \times a - 7 \times 2^{n+1} \\ &= 7 \times (9a - 2^{n+1}) \end{aligned}$$

$\square$

But… $9^2 - 2^3 = 73 = 7 \times 10 + 3$… What went wrong?

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Is the plane just a line?

What's wrong with the following induction proof?

Theorem

For $n \in \mathbb{N}$ greater than or equal to 2, n points of the plane are always aligned.

Proof:

- Base case: when $n = 2$ the property is obviously true.
- Induction step: we assume that the property is true for some $n \geq 2$ and we want to show that it also holds for $n + 1$. Let $A_1, A_2, \dots, A_{n+1}$ be $n + 1$ points of the plane. By the induction hypothesis, we have
  - $A_1, A_2, \dots, A_n$ are on the same line D.
  - $A_2, A_3, \dots, A_{n+1}$ are on the same line D'.

  Then $A_2, A_3, \dots, A_n$ are at the same time on D and D' so that $D = D'$.

  Thus $A_1, \dots, A_{n+1}$ are on the same line. $\square$

Ok, ok… but is there any good induction proof?

Yes, of course! You can try to prove the following statements using induction:

1. For n greater than or equal to 1,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

2. For n greater than or equal to 1, the sum of the n first odd numbers equals n^2.
3. $\forall n \in \mathbb{N}$, 7 divides $9^n - 2^n$.