University of Toronto – MAT137Y1 – LEC0501

Calculus!

Recap of the notations introduced to define integrals

Jean-Baptiste Campesato

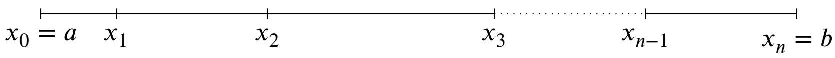
January 16th, 2019

Disclaimer: the goal of these notes is to gather in one place the notations used in this chapter .
They don't replace the videos or the lectures. Particularly, they don't contain any example or any explanation. They are not enough to understand the new concepts of this chapter.
They are just a recap of useful notations and definitions.
As usual, send me an e-mail if you find something wrong/suspicious and I will update the notes.

Definition 1. A **partition** P of the interval $[a, b]$ consists in breaking $[a, b]$ into finitely many closed subintervals. We simply describe it by giving the boundaries of the subintervals:

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

Hence P is a finite set of points of $[a, b]$ containing the endpoints a and b.



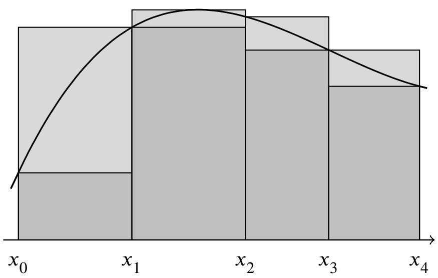
Definition 2. Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$.
We define the **upper Darboux sum** of f with respect to P (or simply the **P-upper sum of f**) by

$$U_P(f) = \sum_{k=1}^{n} \left((x_k - x_{k-1}) \sup_{[x_{k-1}, x_k]} f \right)$$

and the **lower Darboux sum** of f with respect to P (or simply the **P-lower sum of f**) by

$$L_P(f) = \sum_{k=1}^{n} \left((x_k - x_{k-1}) \inf_{[x_{k-1}, x_k]} f \right)$$

In the following figure, the upper Darboux sum is the area of the light grey and the dark grey rectangles together whereas the lower Darboux sum is the area of the dark grey rectangles only.



Remark 3. Notice that the assumption "f is bounded" ensures that the Darboux sums are well-defined. Indeed, then the infimum and the supremum on the subintervals exist thanks to the LUB and GLB principles.

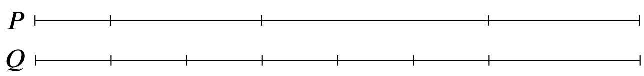
Proposition 4. *For any partition P of $[a,b]$, we have $U_P(f) \geq L_P(f)$.*

Proof.

$$\begin{aligned} U_P(f) &= \sum_{k=1}^{n} \left((x_k - x_{k-1}) \sup_{[x_{k-1},x_k]} f \right) \\ &\geq \sum_{k=1}^{n} \left((x_k - x_{k-1}) \inf_{[x_{k-1},x_k]} f \right) \\ &= L_P(f) \end{aligned}$$

■

Definition 5. Let P and Q be two partitions of $[a,b]$. We say that Q is finer than P if $P \subset Q$.



Proposition 6. *If Q is finer than P then*

$$U_Q(f) \leq U_P(f)$$

and

$$L_Q(f) \geq L_P(f)$$

Proof. By induction, it is enough to see what happens if we break one subinterval into two subintervals. I am just doing it for the upper sum.
Let $c \in (x_{k-1}, x_k)$. Then

$$\begin{aligned} (x_k - x_{k-1}) \sup_{[x_{k-1},x_k]} f &= (x_k - c + c - x_{k-1}) \sup_{[x_{k-1},x_k]} f \\ &= (c - x_{k-1}) \sup_{[x_{k-1},x_k]} f + (x_k - c) \sup_{[x_{k-1},x_k]} f \\ &\geq (c - x_{k-1}) \sup_{[x_{k-1},c]} f + (x_k - c) \sup_{[c,x_k]} f \end{aligned}$$

(Re)watch video 7.6 around 2:20 for a graphical explanation! *Do it!* ■

Proposition 7. *For any partitions P and Q of $[a,b]$, we have $L_P(f) \leq U_Q(f)$.*

Proof. Indeed, set $R = P \cup Q$ then R is finer than P and finer than Q, so

$$L_P(f) \leq L_R(f) \leq U_R(f) \leq U_Q(f)$$

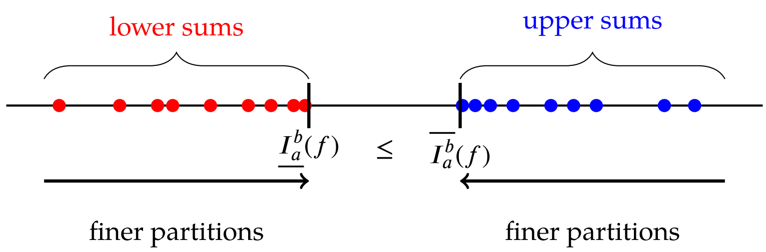
■

Definition 8. Let $f : [a,b] \to \mathbb{R}$ be a bounded function.
We define the **lower integral** of f by

$$\underline{I_a^b}(f) = \sup \left\{ L_P(f),\ \forall P \text{ partition of } [\text{a,b}] \right\}$$

and the **upper integral** of f by

$$\overline{I_a^b}(f) = \inf \left\{ U_P(f),\ \forall P \text{ partition of } [\text{a,b}] \right\}$$



Definition 9. Let $f : [a, b] \to \mathbb{R}$ be a bounded function.
We say that f is **integrable** on $[a, b]$ if $\underline{I}_a^b(f) = \overline{I}_a^b(f)$.
Then we denote this quantity by

$$\int_a^b f(x)dx$$

Theorem 10. *Let f be a bounded function on $[a, b]$.*
Then f is integrable on $[a, b]$ if and only if

$$\forall \varepsilon > 0, \exists \text{ a partition } P \text{ of } [a, b], U_P(f) - L_P(f) < \varepsilon$$

Proof. See lecture (Jan 14). ∎

In the two following theorems, the functions are necessarily bounded (*Can you see why?*). Hence it is not necessary to check it!

Theorem 11. *If $f : [a, b] \to \mathbb{R}$ is non-decreasing or non-increasing then f is integrable on $[a, b]$.*

Proof. See lecture (Jan 14). ∎

Theorem 12. *If $f : [a, b] \to \mathbb{R}$ is continuous then f is integrable on $[a, b]$.*

Definition 13. Let $f : [a, b] \to \mathbb{R}$ be a bounded function.
Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$.
For any $k = 1, 2, \dots, n$, pick a point $x_k^* \in [x_{k-1}, x_k]$.
Then the following sum

$$S_P^*(f) = \sum_{k=1}^{n} \left((x_k - x_{k-1}) f(x_k^*)\right)$$

is called *a* **Riemann sum** of f with respect to P.

Remark 14. Quite often (but not always!), we will pick an endpoint of the subintervals $[x_{k-1}, x_k]$:

1. If for all k, we fix $x_k^* = x_k$, then we talk about the right Riemann sum of f with respect to P.
2. If for all k, we fix $x_k^* = x_{k-1}$, then we talk about the left Riemann sum of f with respect to P.

Definition 15. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$.
The **norm** of P is the length of the longest subinterval of P:

$$\|P\| = \max\left\{x_1 - x_0,\ x_2 - x_1,\ \dots, x_n - x_{n-1}\right\}$$

Theorem 16. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function.*
Let $S_{P_1}^*(f), S_{P_2}^*(f), \dots, S_{P_n}^*(f), \dots$ be a sequence of Riemann sums such that $\lim\limits_{n\to+\infty} \|P_n\| = 0$.
If f is integrable on $[a, b]$ then

$$\int_a^b f(x)dx = \lim_{n\to+\infty} S_{P_n}^*(f)$$

Remark 17. Quite often (but not always!), we will take P_n as the partition breaking $[a, b]$ into n subintervals of the same length.