University of Toronto – MAT137Y1 – LEC0501

Calculus!

A proof of the circle problem

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Disclaimer: this note outreaches the content of the course. You are **NOT** required to read/understand it. It is not part of the course. I only write it to satisfy the curiosity of those who would like to have a proof of the result stated during the lecture (see slide 8) ⋆ .

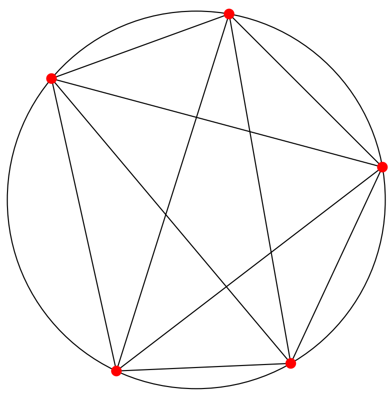
1 Statement

Theorem 1.1. *Put n points on a circle and then join each pair of these points by a line segment. We assume that inside the circle, three of these line segments never intersect at the same point. Then the number of regions obtained inside the circle is*

$$\binom{n-1}{0}+\binom{n-1}{1}+\binom{n-1}{2}+\binom{n-1}{3}+\binom{n-1}{4}$$

which may also be written as the following polynomial

$$\frac{n}{24}(n^3-6n^2+23n-18)+1$$



2 Combinations

Definition 2.1. A k-combination among n elements is a choice of k distinct elements, where the order doesn't matter. We denote by $\binom{n}{k}$ the number of k-combinations among n elements.

⋆ A more selfish reason is that I enjoyed writing this note.

Example 2.2. Let's say that we start with a bag containing four balls which are respectively red, blue, green and yellow : {●, ●, ●, ●}, so that $n = 4$ in this example.

(i) ●● and ●● determine the same 2-combination since the order doesn't matter.

(ii) There are six 2-combinations which are : ●●,●●,●●,●●,●●,●●.
So that $\binom{4}{2} = 6$.

Proposition 2.3. *We have the following equality*

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{if } 0 \leq k \leq n \\ 0 & \text{if } k > n \end{cases}$$

where $n! = 1 \times 2 \times \cdots \times n$ *is the factorial notation (and* $0! = 1$*).*

Proof. If $k > n$ then there is no k-combination among n elements, so that the equality is true.

We now assume that $0 \leq k \leq n$. We pick a first element, so that we have n choices for it, then we have to pick another one among the $n-1$ remaining elements and so on until the k-th element where we have $n-k+1$ elements left. So that we have $n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$ possibilities.

However, different drawings as above may define the same k-combination since we said that the order doesn't matter. To fix this issue, we have to count how many drawings as above are possible for k fixed elements. We first choose an element among our k elements, then another one among the $k-1$ elements, and so one, until the last one where we have only one element left. That means that we may gather all the drawings obtained as above in packets of $k \times (k-1) \times \cdots \times 1 = k!$ elements defining the same k-combination.

Hence the number of k-combinations among n elements is

$$\frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{k!(n-k)!}$$

■

Using the above proposition we may easily check that

$$\binom{n}{k} = \binom{n}{n-k}$$

and

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k} \tag{1}$$

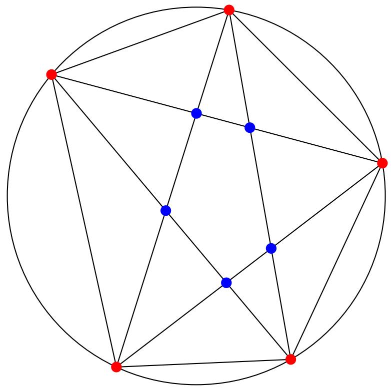
This last equality is quite interesting since it allows us to compute the numbers $\binom{n}{k}$ by filling an array using the following simple rules:

1. Fill the first column with 1.
2. Fill the remaining cases of the first line with 0.
3. Fill the remaining cases line by line using (1): to fill a case, take the element above and sum it with the element directly on it left.

n \ k	0	1	2	3	4	5	⋯
0	1	0	0	0	0	0	
1	1	1	0	0	0	0	
2	1	2	1	0	0	0	
3	1	3	3	1	0	0	
4	1	4	6	4	1	0	
5	1	5	10	10	5	1	
⋮							

The non-zero elements of this array form the famous *Pascal's triangle*.

3 Proof of the theorem



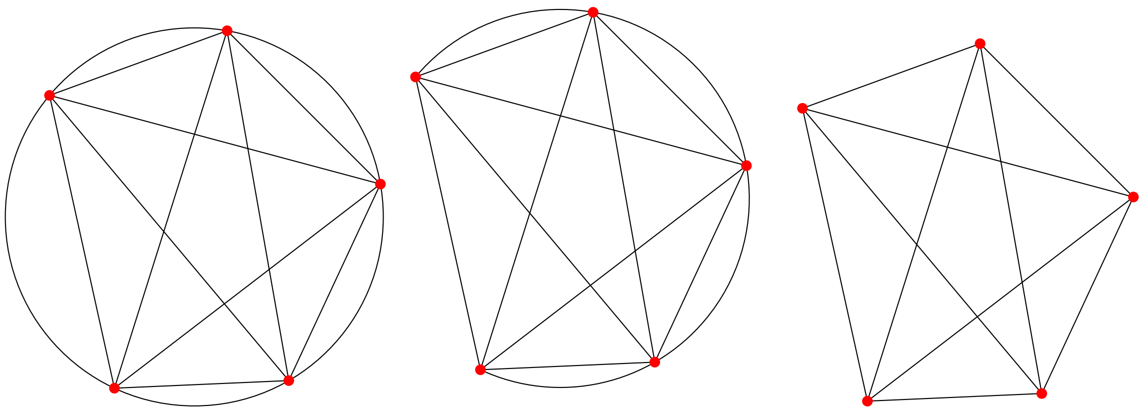
For a figure as in the statement with n points, see above, we denote by :

- $V(n)$ the number of vertices (a vertex is either the intersection of two line segments inside the circle, or a point of the beginning, in red in the figure).
- $E(n)$ the number of edges (an edge is a line linking two vertices, including the arcs between the red vertices).
- $R(n)$ the number of regions (this is the number that we would like to compute).

Lemma 3.1. *Regardless of n and the position of the points on the circle, the following equality always holds*

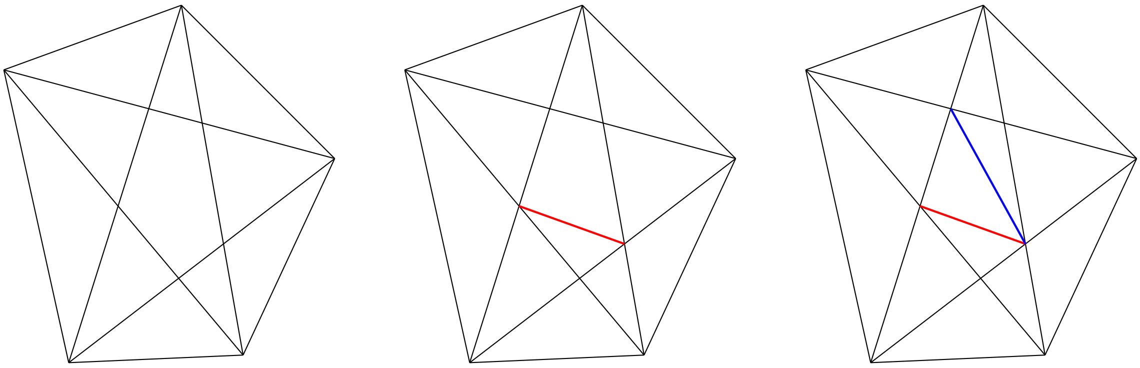
$$V(n) - E(n) + R(n) = 1$$

Proof. ⋆
Step 1: the quantity $V(n) - E(n) + R(n)$ remains unchanged if we remove an arc.
Indeed, if we remove an arc, we remove one edge and one region (see the figure below) so that $E(n)$ becomes $E(n)-1$ and $R(n)$ becomes $R(n)-1$, but then $V(n)-(E(n)-1)+(R(n)-1) = V(n)-E(n)+R(n)$.
So we may remove the arcs one by one without changing the quantity $V(n) - E(n) + R(n) = 1$.



Step 2: the quantity $V(n) - E(n) + R(n)$ remains unchanged if we add an edge joining two non-adjacent vertices of a region which is not a triangle. Indeed, in this case $E(n)$ becomes $E(n)+1$ and $R(n)$ becomes $R(n) + 1$ (since we divided a region into two regions, see the figure below).
But then $V(n) - (E(n) + 1) + (R(n) + 1) = V(n) - E(n) + R(n)$.
So we may assume that all the regions are triangles.

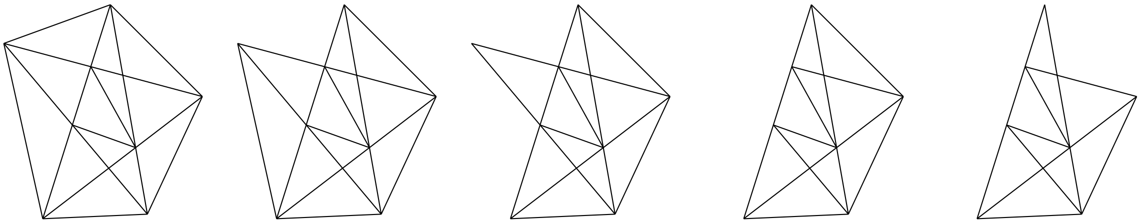
⋆ This proof is closely related to a proof of the famous Euler formula for polyhedra: $V - E + F = 2$.



Step 3: the quantity $V(n) - E(n) + R(n)$ remains unchanged if we remove a boundary triangle. Be careful, different situations occur:

(i) either E and R both decrease by one (if the triangle is *connected* along two edges)

(ii) or V decreases by one, E by two and R by one (if the triangle is *connected* along one edge).



Hence, as in the above example, we may remove such triangles one by one without changing the quantity $V(n) - E(n) + R(n)$. We repeat this process until we have only one triangle left.
Finally, for a triangle, $V - E + R = 3 - 3 + 1 = 1$. ■

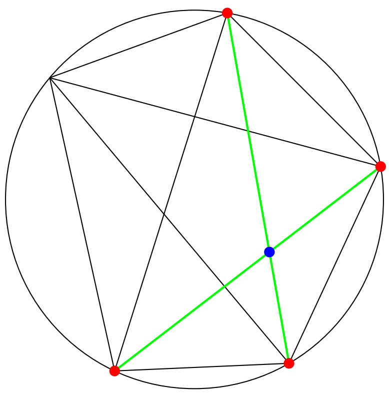
According to the previous lemma, it is enough to compute $V(n)$ and $E(n)$ in order to get $R(n)$. Now, the question is: are these two numbers easier to compute? And the answer is... yes!

Lemma 3.2. $V(n) = n + \binom{n}{4}$

Proof. There are two kinds of vertices:

(i) The initial points on the circle, of which there are n.

(ii) The intersection points inside the circle.



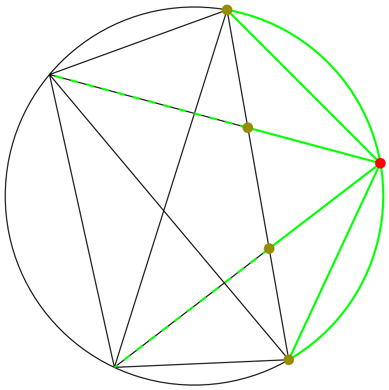
But an intersection point is given by the intersection of exactly two line segments (remember that we banned the intersection of three or more line segments at the same point), and a line segment is given by two points on the circle. To sum up, an intersection point corresponds to four points on the circle. Hence there are $\binom{n}{4}$ points of this kind.

■

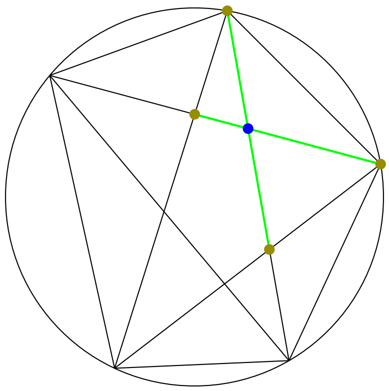
Lemma 3.3. $E(n) = \frac{n(n-1)}{2} + 2\binom{n}{4}$

Proof. Instead of counting directly the number of edges, the trick here is to compute the number of extremities of edges (and then to divide this number by 2). Indeed, it is easier to count how many edges come from a given vertex. But, again, we have to distinguish the two kinds of vertices.

(i) We first focus on the initial points on the circle. Such a point is the extremity of $n+1$ edges (don't forget the arcs of the circle, which are edges!). And there are n such points.
So we get $n(n+1)$.



(ii) Now, we focus on the intersection points inside the circle. Such a point is the extremity of 4 edges. And, as computed before, there are $\binom{n}{4}$ such points.
So we get $4\binom{n}{4}$.



During these two steps, we counted $n(n+1) + 4\binom{n}{4}$ edges. But each edge has two extremities, so it has be counted twice. Thus we have to divide $n(n+1) + 4\binom{n}{4}$ by 2 to conclude. ∎

We have now everything to conclude.

Proof of Theorem 1.1.
From Lemma 3.1, we have $R(n) = E(n) - V(n) + 1$. Then, using Lemma 3.2 and Lemma 3.3, we get

$$\begin{aligned} R(n) &= E(n) - V(n) + 1 \\ &= \frac{n(n+1)}{2} + 2\binom{n}{4} - n - \binom{n}{4} + 1 \\ &= \frac{n^2+n}{2} - n + \binom{n}{4} + 1 \qquad (2) \\ &= \frac{n^2+n-2n}{2} + \frac{n!}{4!(n-4)!} + 1 \\ &= \frac{n^2-n}{2} + \frac{n(n-1)(n-2)(n-3)}{24} + 1 \\ &= \frac{12n^2-12n}{24} + \frac{n^4-6n^3+11n^2-6n}{24} + 1 \\ &= \frac{n^4-6n^3+23n^2-18n}{24} + 1 \end{aligned}$$

The other form derives from (2). Indeed, by the formula (1), we have

- $\binom{n}{4} = \binom{n-1}{4} + \binom{n-1}{3}$
- $\frac{n^2+n}{2} - n = \frac{n^2-n}{2} = \frac{n(n-1)}{2} = \frac{n!}{2!(n-2)!} = \binom{n}{2} = \binom{n-1}{2} + \binom{n-1}{1}$

and $1 = \binom{n-1}{0}$.
Thus

$$\begin{aligned} R(n) &= \frac{n^2+n}{2} - n + \binom{n}{4} + 1 \\ &= \binom{n-1}{2} + \binom{n-1}{1} + \binom{n-1}{4} + \binom{n-1}{3} + \binom{n-1}{0} \end{aligned}$$

∎

4 And…what about the powers of 2?

During the lecture, we empirically noticed that $R(1) = 1$, $R(2) = 2$, $R(3) = 4$, $R(4) = 8$, $R(5) = 16$…and then that $R(6) = 31$…
Hence a natural question is *"Why do we first obtain the powers of 2?"*
That's because of Newton's binomial formula :

$$2^{n-1} = (1+1)^{n-1} = \binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{n-1}$$

Indeed, this expansion coincides with our formula for $R(n)$, as soon as $n - 1 \leq 4$ (or equivalently, as soon as $n \leq 5$).
For example: $R(3) = \binom{2}{0} + \binom{2}{1} + \binom{2}{2} + \binom{2}{3} + \binom{2}{4} = \binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 2^2$.