MAT 137Y: Calculus!
Problem Set 10
Due on TUESDAY, April 2 by 11:59pm via crowdmark

1. Consider the power series

$$S(x) = \sum_{n=r}^{\infty} \frac{P(n)}{Q(n)} \left(\frac{x-c}{a} \right)^n$$

where P and Q are polynomials, $r \in \mathbb{N}$, $a \in \mathbb{R}$, $c \in \mathbb{R}$. Assume $a \neq 0$. Assume P is not the zero polynomial. Assume for every $n \in \mathbb{N}$, $Q(n) \neq 0$.

Find the radius of convergence and the interval of convergence of the power series S.

We know that P and Q are non-zero polynomials, so they can be written as

$$P(n) = \sum_{i=0}^{d} \alpha_i n^i \qquad \text{and} \qquad Q(n) = \sum_{i=0}^{e} \beta_i n^i$$

where $\alpha_d \neq 0$ and $\beta_e \neq 0$.
We say that d is the degree of P (resp. e is the degree of Q) and that α_d is the leading coefficient of P (resp. β_e is the leading coefficient of Q).

The radius of convergence is $|a|$.

- If $e \leq d$ then the interval of convergence is $\big(c - |a|, c + |a|\big)$.
- If $e = d + 1$ and $a > 0$ then the interval of convergence is $\big[c - a, c + a\big)$.
- If $e = d + 1$ and $a < 0$ then the interval of convergence is $\big(c + a, c - a\big]$.
- If $e \geq d + 2$ then the interval of convergence is $\big[c - |a|, c + |a|\big]$.

Useful facts. Let $P(x) = \sum_{i=0}^{d} \alpha_i x^i$ be a polynomial with $d \in \mathbb{N}_{\geq 0}$ and $\alpha_d \neq 0$.

(a) Then P has at most d roots:
Using division of polynomials, it is easy to check that

$$P(x_0) = 0 \Leftrightarrow P(x) = (x - x_0)\tilde{P}(x)$$

where $\tilde{P}$ is a polynomial of degree $\deg(P) - 1$.
Hence the number of roots can't exceed the degree.

(b) *(Lagrange's bound)* If $P(x_0) = 0$ then $|x_0| \leq \max\left(1, \sum_{i=0}^{d-1} \left|\frac{a_i}{a_d}\right|\right)$.

Indeed assume that $P(x_0) = 0$ and that $|x_0| \geq 1$, then $a_d x_0^d = -\sum_{i=0}^{d-1} a_i x_0^i$ and hence

$$|a_d x_0^d| \leq \sum_{i=0}^{d-1} |a_i x_0^i| \leq \sum_{i=0}^{d-1} |a_i||x_0|^{d-1}$$

dividing by $|a_d||x_0|^{d-1}$ we get

$$|x_0| \leq \sum_{i=0}^{d-1} \left|\frac{a_i}{a_d}\right|.$$

Let's start answering the question.
We begin by applying the ratio test when $x \neq c$ (notice that $P(n) \neq 0$ for n large enough, by using any one of the two above useful facts).

$$\lim_{n\to+\infty} \left| \frac{\frac{P(n+1)}{Q(n+1)} \left(\frac{x-c}{a}\right)^{n+1}}{\frac{P(n)}{Q(n)} \left(\frac{x-c}{a}\right)^{n}} \right| = \lim_{n\to+\infty} \left| \frac{P(n+1)}{P(n)} \cdot \frac{Q(n)}{Q(n+1)} \cdot \frac{x-c}{a} \right| = \left| \frac{x-c}{a} \right|$$

The last equality comes from the fact that $\tilde{P}(n) = P(n+1)$ is a polynomial with the same degree and the same leading coefficient as P (just write $P(n+1)$ and expand it). The same claim holds for $Q(n+1)$.

Hence, according to the ratio test,

- if $|x-c| < |a|$ then $S(x)$ is convergent.
- if $|x-c| > |a|$ then $S(x)$ is divergent.

Furthermore, the radius of convergence of S is $|a|$.
To determine the interval of convergence, we still have to study the case $|x-c| = |a|$.

We will need the following two lemmas:

Lemma 1. *Let* $P(x) = \sum_{i=0}^{d} \alpha_i x^i$ *be a polynomial with* $d \in \mathbb{N}_{\geq 0}$ *and* $\alpha_d \neq 0$.
If the leading coefficient α_d *of* P *is positive then* P *is eventually positive.*
If the leading coefficient α_d *of* P *is negative then* P *is eventually negative.*

Proof. If $d = 0$, the polynomial is constant and there is nothing to show.
Hence, we may assume that $d > 0$.
Notice that

$$\begin{aligned}\lim_{x\to+\infty} P(x) &= \lim_{x\to+\infty} \sum_{i=0}^{d} \alpha_i x^i \\ &= \lim_{x\to+\infty} \alpha_d x^d \left(1 + \sum_{i=0}^{d-1} \frac{\alpha_i}{\alpha_d x^{d-i}}\right) \\ &= \begin{cases} +\infty & \text{when } \alpha_d > 0 \\ -\infty & \text{when } \alpha_d < 0 \end{cases}\end{aligned}$$

By the very definition of these limits, we get in the first case that

$$\exists A \in \mathbb{R}, \forall x \in \mathbb{R}, (x > A \implies P(x) > 0)$$

and in the second case that

$$\exists A \in \mathbb{R}, \forall x \in \mathbb{R}, (x > A \implies P(x) < 0)$$

□

Lemma 2. *A rational function* $f(x) = \frac{P(x)}{Q(x)}$, *where* P *and* Q *are two polynomials and* $Q \neq 0$, *is eventually monotonic.*

Proof. Using any one of the two above useful facts, it is easy to notice that f is well-defined for x large enough (since $Q(x) \neq 0$ for x large enough).
Notice that f is differentiable on its domain as a rational function and that

$$f'(x) = \frac{P'(x)Q(x) - P(x)Q'(x)}{Q(x)^2}$$

Since the numerator is a polynomial, we conclude using Lemma 1 that f' is eventually non-negative or eventually non-positive.
So f is eventually non-decreasing or non-increasing. □

Let's come back to the question.

(a) First case: $x - c = a$. Then $S(c + a) = \sum_{n=r}^{+\infty} \frac{P(n)}{Q(n)}$.

Up to multiplying S by -1 (which doesn't change the convergence or divergence of the series), we may assume that $\alpha_d > 0$, $\beta_e > 0$ and, using Lemma 1, $\frac{P(n)}{Q(n)} > 0$ for n large enough.

Notice that $\lim_{n\to+\infty} \frac{\frac{P(n)}{Q(n)}}{\frac{1}{n^{e-d}}} = \lim_{n\to+\infty} \frac{\frac{P(n)}{n^d}}{\frac{Q(n)}{n^e}} = \frac{\alpha_d}{\beta_e} > 0.$

Hence, using the LCT, $S(c+a)$ is convergent if and only $\sum_{n=r}^{\infty} \frac{1}{n^{e-d}}$ is convergent.

But we know that the latter is convergent if and only if $e - d > 1$.

(b) Second case: $x - c = -a$. Then $S(c-a) = \sum_{n=r}^{+\infty} (-1)^n \frac{P(n)}{Q(n)}$.

i. If $d \geq e$ then $\lim_{n\to+\infty} (-1)^n \frac{P(n)}{Q(n)} \neq 0$.
(notice that $\lim_{n\to+\infty} a_n = 0 \Leftrightarrow \lim_{n\to+\infty} |a_n| = 0$, so we don't have to take care of $(-1)^n$ to check the above limit).
Hence $S(c-a)$ is divergent.

ii. Assume that $d < e$ and set $b_n = \frac{P(n)}{Q(n)}$.
Applying Lemma 1 to P and Q, we know that b_n is eventually of constant sign. Hence, up to multiplying S by -1 if necessary, we may assume that $b_n > 0$ for n large enough.
Then we know that

- b_n is eventually positive
- $\lim_{n\to+\infty} b_n = 0$
- b_n is eventually non-increasing: by Lemma 2, we know that b_n is eventually monotonic but since b_n is eventually positive and tends to 0, it has to be eventually non-increasing.

Using the AST, we deduce that $S(c-a) = \sum_{n=r}^{+\infty} (-1)^n b_n$ is convergent.

2. (a) Let $f(x) = e^{1/x}$. Use the Maclaurin series of $g(x) = e^x$ to write f as a "power series with negatives exponents". For which values of x is $f(x)$ actually equal to this series?

From the Maclaurin series of g, we obtain

$$\sum_{n=0}^{+\infty} \frac{\left(\frac{1}{x}\right)^n}{n!} = \sum_{n=0}^{+\infty} \frac{x^{-n}}{n!}$$

We know that $g(x) = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$.

Hence $f(x) = \sum_{n=0}^{+\infty} \frac{x^{-n}}{n!}$ for all $x \in \mathbb{R} \setminus \{0\}$.

(b) Let $h(x) = xe^{1/x}$. Use your answer to Question 2a (write down the first few terms to see what it looks like) to find the slant asymptote of the function h.

We know that $h(x) = \sum_{n=0}^{+\infty} \frac{x^{-n+1}}{n!} = x + 1 + \sum_{n=2}^{+\infty} \frac{x^{-n+1}}{n!}$.

So the slant asymptote is $y = x + 1$.

3. Let $a, b, c \in \mathbb{R}$. Let $k, N \in \mathbb{N}$. We define the function

$$H(x) = \int_{ax}^{bx} x^k e^{-c^2 t^2}\, dt.$$

Find a formula for $H^{(N)}(0)$.

If $N \geq k+1$ and $N-k$ is odd then

$$H^{(N)}(0) = \frac{(-1)^{\frac{N-k-1}{2}} c^{N-k-1} \left(b^{N-k} - a^{N-k}\right)(N!)}{(N-k)\left(\frac{N-k-1}{2}\right)!}$$

Otherwise, $H^{(N)}(0) = 0$.

$$\begin{aligned}
H(x) &= \int_{ax}^{bx} x^k e^{-c^2 t^2}\, dt \\
&= \int_{ax}^{bx} x^k \sum_{n=0}^{+\infty} \frac{(-1)^n c^{2n}}{n!} t^{2n}\, dt \\
&= \sum_{n=0}^{+\infty} (-1)^n \frac{c^{2n}}{n!} x^k \int_{ax}^{bx} t^{2n}\, dt \\
&= \sum_{n=0}^{+\infty} (-1)^n \frac{c^{2n}}{n!} x^k \frac{b^{2n+1} - a^{2n+1}}{2n+1} x^{2n+1} \\
&= \sum_{n=0}^{+\infty} \frac{(-1)^n c^{2n} \left(b^{2n+1} - a^{2n+1}\right)}{(2n+1)(n!)} x^{2n+k+1} \\
&= \sum_{N=0}^{+\infty} \frac{H^{(N)}(0)}{N!} x^N
\end{aligned}$$