MAT 137Y: Calculus!
Problem Set 9
Due on Thursday, March 14 by 11:59pm via crowdmark

1. (a) Let $a \in \mathbb{R}$. For which values of a is the integral

$$\int_1^2 \frac{1}{(x-1)^a}\,dx$$

convergent? For which is it divergent?

Box your final answer at the top of your solution, then proceed to explain, compute, or prove anything you need to.

$$\boxed{\int_1^2 \frac{1}{(x-1)^a}dx \text{ is convergent if and only if } a < 1.}$$

Proof :

Since $f(x) = \frac{1}{(x-1)^a}$ is continuous on $(1,2]$, the integral is only improper at 1.

Let $t \in (1,2]$, then,

$$\int_t^2 \frac{1}{(x-1)^a}dx = \int_{t-1}^1 \frac{1}{u^a}du$$

using the substitution $u = x - 1$.

But we already know (Riemann's improper integral $\frac{1}{x^a}$ at 0) that $\lim\limits_{t\to 1^+} \int_{t-1}^1 \frac{1}{u^a}du$ exists if and only if $a < 1$.

Hence $\lim\limits_{t\to 1^+} \int_t^2 \frac{1}{(x-1)^a}dx$ exists if and only if $a < 1$. Q.E.D.

(b) Let $a, b, c \in \mathbb{R}$. For which values of a, b, and c is the integral

$$\int_1^2 \frac{e^{cx}}{(x-1)^a(\ln x)^b}\,dx.$$

convergent? For which values is it divergent?

Box your final answer at the top of your solution, then proceed to explain, compute, or prove anything you need to.

Hint: Calculate $\lim_{x\to 1} \frac{\ln x}{x-1}$. Then attempt the problem when $a = c = 0$ first.

$$\boxed{\int_1^2 \frac{e^{cx}}{(x-1)^a(\ln x)^b}dx \text{ is convergent if and only if } a+b<1.}$$

The proof relies on the following lemma:

Lemma. $\lim_{x\to 1} \frac{x-1}{\ln x} = 1$

Proof. We know that:

- $\lim_{x\to 1}(x-1) = 0$ and $\lim_{x\to 1}\ln(x) = 0$.
- $x \mapsto (x-1)$ and ln are differentiable near 1.
- $\ln(x)$ and $\ln'(x) = \frac{1}{x}$ don't vanish near 1 (except at 1 for ln).
- $\lim_{x\to 1} \frac{\frac{d(x-1)}{dx}}{\frac{d\ln(x)}{dx}} = \lim_{x\to 1} \frac{1}{\frac{1}{x}} = 1$

Hence, according to L'Hôpital's rule, $\lim_{x\to 1} \frac{x-1}{\ln x} = 1$. □

Proof :

Since $g(x) = \frac{e^{cx}}{(x-1)^a(\ln x)^b}$ is continuous on $(1, 2]$, the integral is improper at 1.

Let $h(x) = \frac{1}{(x-1)^{a+b}}$ then

i. g and h are continuous on $(1, 2]$,

ii. $\forall x \in (1, 2]$, $g(x) \geq 0$,

iii. $\forall x \in (1, 2]$, $h(x) > 0$, and,

iv. $\lim_{x\to 1^+} \frac{g(x)}{h(x)} = \lim_{x\to 1^+} e^{cx}\left(\frac{x-1}{\ln x}\right)^b = e^c \times 1 = e^c > 0$ (using the above lemma).

Hence, according to the LCT, $\int_1^2 \frac{e^{cx}}{(x-1)^a(\ln x)^b}dx$ is convergent if and only if $\int_1^2 \frac{1}{(x-1)^{a+b}}dx$ is convergent.

According to the first question, the latter improper integral is convergent if and only if $a+b<1$. Q.E.D.

2. Let f be a **bounded, continuous, non-negative** function defined, at least, on an interval of the form $[c, \infty)$ for some $c \in \mathbb{R}$.

   (a) Prove that IF $\lim_{x\to\infty} f(x)$ exists and is not 0, THEN the improper integral $\int_c^\infty f(x)dx$ is divergent.

   Hint: To get a feel for this question, draw a picture of f with $\lim_{x\to\infty} f(x) = 0.1$, and convince yourself why it's true in this case. Then write the general proof. You will need to use the definition of $\lim_{x\to\infty} f(x)$. Your argument will also probably involve the BCT.

   We know that there exists $l \in \mathbb{R} \setminus \{0\}$ such that $\lim_{x\to+\infty} f(x) = l$, i.e.

$$\forall \varepsilon > 0, \exists A \in \mathbb{R}, \forall x \in [c, +\infty), (x \geq A \implies |f(x) - l| < \varepsilon) \tag{1}$$

   Lemma. $l > 0$

   Sketch of proof. Assume by contradiction that $l < 0$.
   Applying (1) to $\varepsilon = -\frac{l}{2} > 0$, we know that for x large enough

$$f(x) - l = |f(x) - l| < -\frac{l}{2}$$

   where the first inequality comes from the fact that $f(x)$ and $-l$ are both non-negative.
   Therefore $f(x) < \frac{l}{2} < 0$, which contradicts the fact that $f(x) \geq 0$.
   Hence $l \geq 0$. But, we also assumed that $l \neq 0$. □

   We want to show that $\int_c^{+\infty} f(x)dx$ is divergent.

   Proof 1:

   - Applying (1) to $\varepsilon = \frac{l}{2} > 0$, we know that there exists $A \in \mathbb{R}$ such that for any $x \in [c, +\infty)$,
$$\text{if } x \geq A \text{ then } |f(x) - l| < \frac{l}{2}. \tag{2}$$
   - Set $d = \max(c, A)$. Let $x \in [d, +\infty)$.
     Therefore $f(x)$ is well-defined and $x \geq A$. Hence, by (2), we deduce that $|f(x) - l| < \frac{l}{2}$ which is equivalent to $-\frac{l}{2} < f(x) - l < \frac{l}{2}$ and from which we derive that $\frac{l}{2} < f(x)$.
   - We know that
     i. $x \mapsto \frac{l}{2}$ and f are continuous on $[d, +\infty)$,

ii. $\forall x \in [d, +\infty)$, $0 \leq \frac{l}{2} \leq f(x)$, and,

iii. $\displaystyle\int_d^{+\infty} \frac{l}{2} dx$ is divergent since $\displaystyle\lim_{t\to+\infty} \int_d^t \frac{l}{2} dx = \lim_{t\to+\infty} \frac{l}{2}t - \frac{l}{2}d = +\infty$.

So, according to the BCT, $\displaystyle\int_d^{+\infty} f(x)dx$ is divergent.

- Hence $\displaystyle\lim_{t\to+\infty} \int_c^t f(x)dx = \lim_{t\to+\infty} \left(\int_c^d f(x)dx + \int_d^t f(x)dx \right)$ doesn't exist.
- We proved that $\displaystyle\int_c^{+\infty} f(x)dx$ is divergent. Q.E.D.

Proof 2:

Notice that $\displaystyle\lim_{x\to+\infty} \frac{f(x)}{1} = l > 0$.

Hence, according to the LCT, $\displaystyle\int_c^{+\infty} f(x)dx$ is divergent since $\displaystyle\int_c^{+\infty} 1dx$ is.

(b) Prove that IF f is eventually decreasing and the improper integral $\displaystyle\int_c^{\infty} f(x)dx$ is convergent, THEN $\displaystyle\lim_{x\to\infty} f(x) = 0$.

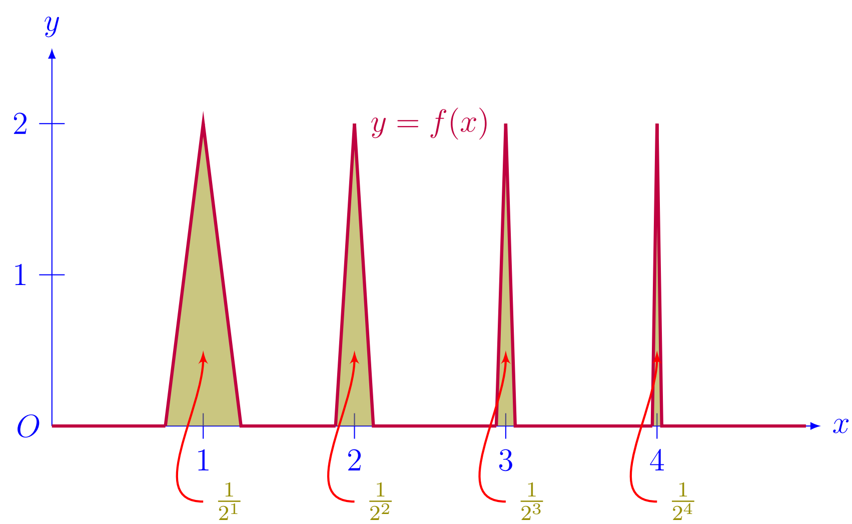
Hint: This is a very short proof. Seriously: if you do not find it very short, you are missing something. Use Question 1 from PS6 and Question 2a from this Problem Set.

We know that f is eventually decreasing and bounded on $[c, +\infty)$, hence, according to Q1 of PS6, $l = \displaystyle\lim_{x\to+\infty} f(x)$ exists.

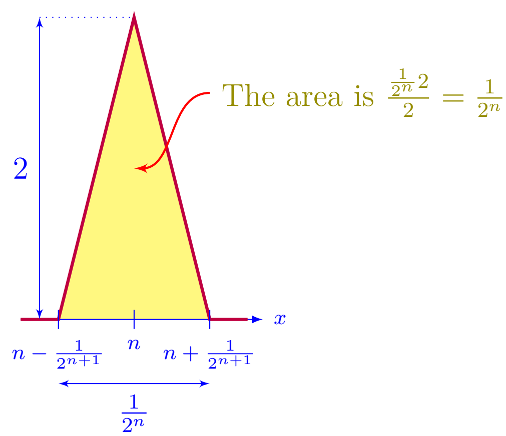
Now, assume by contradiction that $l \neq 0$.
Then, according to the previous question, $\int_c^{+\infty} f(x)dx$ is divergent.
Which contradicts our assumption that this improper integral is convergent.
Conclusion: $\displaystyle\lim_{x\to+\infty} f(x) = 0$. Q.E.D.

(c) Show with an example that it is possible that the improper integral $\int_c^\infty f(x)dx$ be convergent while $\lim_{x\to\infty} f(x)$ is not 0.

Consider the function f defined on $[0, +\infty)$ whose graph consists into triangular peaks centered around each $n \in \mathbb{N}_{>0}$ with base length $\frac{1}{2^n}$ and height 2 (and the function is equal to 0 between the peaks).



Here is a zoom around $x = n \in \mathbb{N} \setminus \{0\}$:



Intuitively, the area below the graph is given by the convergent geometric series $\sum_{n=1}^{+\infty} \frac{1}{2^n}$ and therefore such a function provides a suitable counter-example.

Let's prove it formally.
First, by construction, this function is bounded, non-negative and continuous.

<u>Claim 1:</u> $\int_0^{+\infty} f(t)dt$ is convergent.

- For $x \in [0, +\infty)$, we set $F(x) = \int_0^x f(t)dt$.
- F is non-decreasing: let $x_1, x_2 \in [0, +\infty)$ such that $x_1 < x_2$ then

$$F(x_2) - F(x_1) = \int_{x_1}^{x_2} f(t)dt \geq 0 \quad (\text{since } x_2 > x_1 \text{ and } f \geq 0)$$

  and hence $F(x_1) \leq F(x_2)$.
- F is bounded from above: let $x \in [0, +\infty)$ and denote by $\lceil x \rceil$ the ceiling of x which is the least integer greater than or equal to x. Then

$$\begin{aligned} F(x) &\leq F(\lceil x \rceil) \\ &= \int_0^{\lceil x \rceil} f(t)dt \\ &= \text{Area below the graph of } f \text{ on } [0, \lceil x \rceil] \\ &= \sum_{n=1}^{\lceil x \rceil} \left(\frac{1}{2}\right)^n \\ &< \sum_{n=1}^{+\infty} \left(\frac{1}{2}\right)^n = 1 \end{aligned}$$

  The last inequality derives from the fact that the sequence of the partial sums $\left(\sum_{n=1}^{k} \frac{1}{2^n}\right)_{k \geq 1}$ is increasing and convergent, hence, from the MCT, its limit is its upper bound.
- According to the MCT for functions, $\lim_{x \to +\infty} F(x)$ exists, which proves the claim.

<u>Claim 2:</u> $\lim_{x \to +\infty} f(x)$ doesn't exist.

- Assume by contradiction that $\lim_{x \to +\infty} f(x) = l$.
- Then, by considering the sequence $(f(n))_{n \geq 1}$, we get that

$$l = \lim_{n \to +\infty} f(n) = \lim_{n \to +\infty} 2 = 2$$

- similarly, for the sequence $\left(f\left(n+\frac{1}{2}\right)\right)_{n\geq 1}$, we obtain

$$l = \lim_{n\to+\infty} f\left(n+\frac{1}{2}\right) = \lim_{n\to+\infty} 0 = 0$$

- Hence, $0 = 2$, which is false.

We have constructed a function $f\colon [0,+\infty) \to \mathbb{R}$ which is continuous, bounded and non-negative, such that $\displaystyle\int_0^{+\infty} f(x)dx$ is convergent whereas $\displaystyle\lim_{x\to+\infty} f(x)$ doesn't exist. Q.E.D.