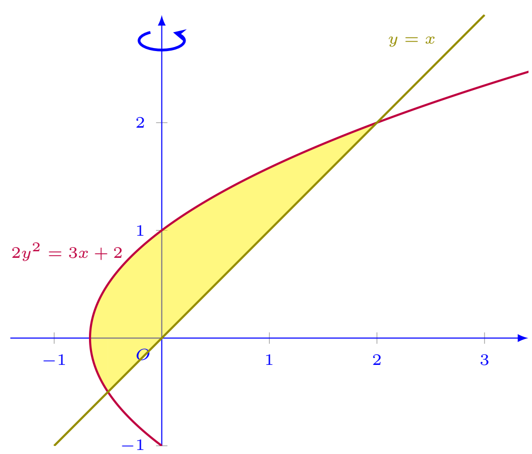
MAT 137Y: Calculus!
Problem Set 8
Due on Thursday, February 28 by 11:59pm via crowdmark

1. Let R be the bounded region enclosed by the curves $y = x$ and $2y^2 = 3x + 2$. We rotate R around the y-axis. Compute the volume of the resulting solid.

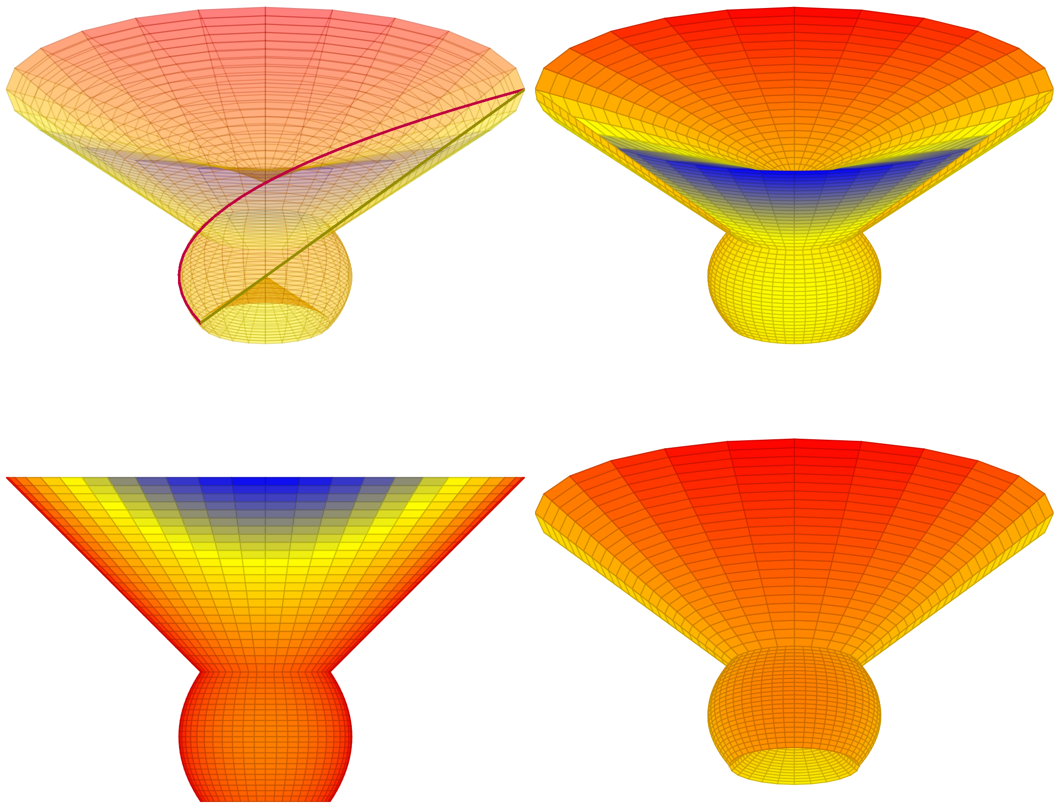
   Notes: Careful! The region R intersects three quadrants. The axis of rotation cuts across the region, and you need to figure out what that does to the solid.

   The difficulty of this problem is the set up. The volume can be written as an integral (or a sum of integrals) of polynomials. Make sure you explain all the process to get it to that form. If you get it to that form, you do not need to perform the integration in detail. You may jump directly from that expression to the final answer.

   Below is a drawing recapitulating the situation: we want to compute the volume of the solid obtained by revolving the yellow region around the y-axis.



Below is the solid obtained by revolving the yellow region around the y-axis, viewed under three different angles:



Since the axis of revolution passes through the yellow region, one has to be very careful not to count any piece of volume twice.
For this purpose, I am first going to check at each level $y = a$, $a \in (0, 1)$, whether the region is larger on the left side or on the right side of the y-axis.
Indeed, the largest region will overlap the smallest one after taking the revolution around the y-axis.

On the right side, the border is delimited by $x = y$ and on the left side the border is delimited by $x = \frac{2}{3}y^2 - \frac{2}{3}$. So we need to find, for each $y \in (0, 1)$, whether

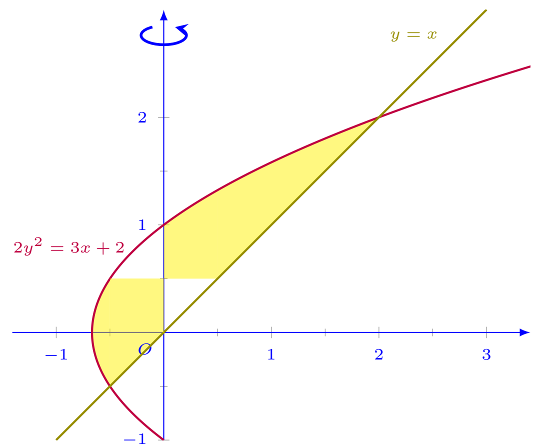
$$|y| \leq \left|\frac{2}{3}y^2 - \frac{2}{3}\right| \quad \text{or} \quad |y| \geq \left|\frac{2}{3}y^2 - \frac{2}{3}\right|$$

or equivalently, since $y \in (0, 1)$, whether

$$y \leq -\frac{2}{3}y^2 + \frac{2}{3} \quad \text{or} \quad y \geq -\frac{2}{3}y^2 + \frac{2}{3}$$

Since $2y^2 + 3y - 2 = (y+2)(2y-1)$, we need to take into account the left side of the yellow region when $y \in \left(0, \frac{1}{2}\right)$ and the right side when $y \in \left(\frac{1}{2}, 1\right)$.

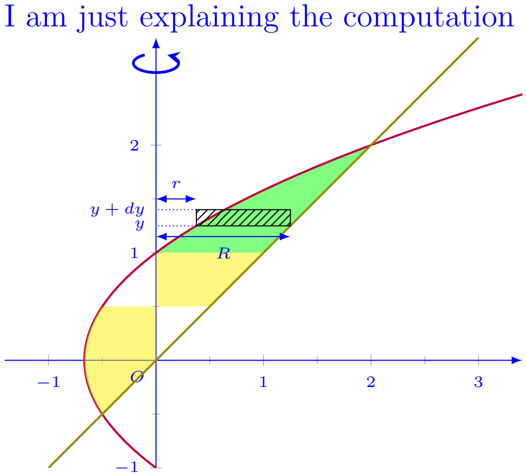
To recap, we obtain the same solid of revolution by using the revolution around the y-axis of the following yellow region, but without any overlapping.

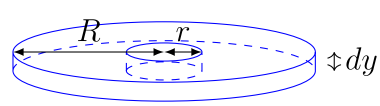


<u>Method 1:</u> using the slicing method.
Since the purple curve is given by $x = \frac{2}{3}y^2 - \frac{2}{3}$, it is easier to integrate with respect to the y-variable (in order to avoid working with square roots).
For this reason, it is a good idea to use the slicing method since with this method we integrate along the axis parallel to the axis of revolution (whereas with the shell method we integrate along the axis perpendicular to the axis of revolution).

We are going to apply the slicing method separately for $y \in \left(-\frac{1}{2}, 0\right)$, $y \in \left(0, \frac{1}{2}\right)$, $y \in \left(\frac{1}{2}, 1\right)$ and $y \in (1, 2)$.
Indeed, for these different values, the regions are not enclosed by the same curves.

I am just explaining the computation for $y \in (1, 2)$, the other cases are similar.





The volume of the slice obtained by revolving the rectangle around the y-axis is

$$dV = (\pi R^2 - \pi r^2)dy = \pi \left(y^2 - \left(\frac{2}{3}y^2 - \frac{2}{3} \right)^2 \right) dy$$

Hence the volume of the solid obtained by revolving the green region around the y-axis is

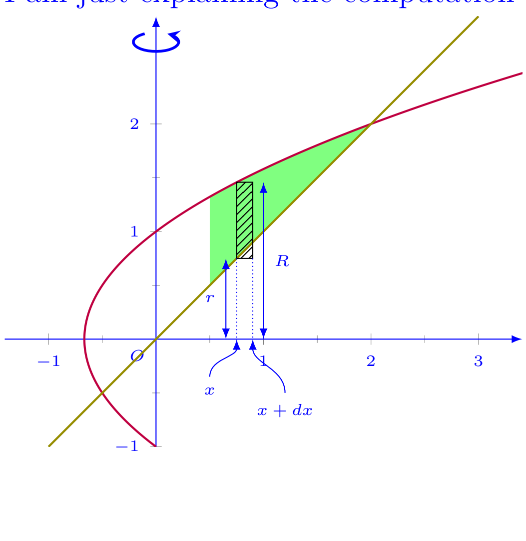
$$\pi \int_1^2 \left(y^2 - \left(\frac{2}{3}y^2 - \frac{2}{3} \right)^2 \right) dy$$

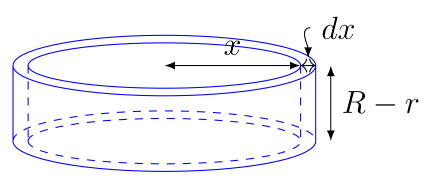
By repeating the above process to the other parts, we obtain that the total volume of the solid is

$$\begin{aligned} V &= \pi \int_{-\frac{1}{2}}^{0} \left(\left(\frac{2}{3}y^2 - \frac{2}{3} \right)^2 - y^2 \right) dy + \pi \int_0^{\frac{1}{2}} \left(\frac{2}{3}y^2 - \frac{2}{3} \right)^2 dy \\ &\quad + \pi \int_{\frac{1}{2}}^{1} y^2 dy + \pi \int_1^2 \left(y^2 - \left(\frac{2}{3}y^2 - \frac{2}{3} \right)^2 \right) dy \\ &= \frac{79}{540}\pi + \frac{203}{1080}\pi + \frac{7}{24}\pi + \frac{163}{135}\pi \\ &= \frac{11}{6}\pi \end{aligned}$$

<u>Method 2:</u> using the shell method.
We are now using the shell method, but applied separately for $x \in \left(-\frac{2}{3}, -\frac{1}{2}\right)$, $x \in \left(-\frac{1}{2}, 0\right)$, $x \in \left(0, \frac{1}{2}\right)$ and $x \in \left(\frac{1}{2}, 2\right)$.

I am just explaining the computation for $x \in \left(\frac{1}{2}, 2\right)$, the other cases are similar.





The volume of the shell obtained by revolving the rectangle around the y-axis is approximately

$$dV \simeq (2\pi x)\cdot(R-r)dx = (2\pi x)\left(\sqrt{\frac{3}{2}x+1}-x\right)dx$$

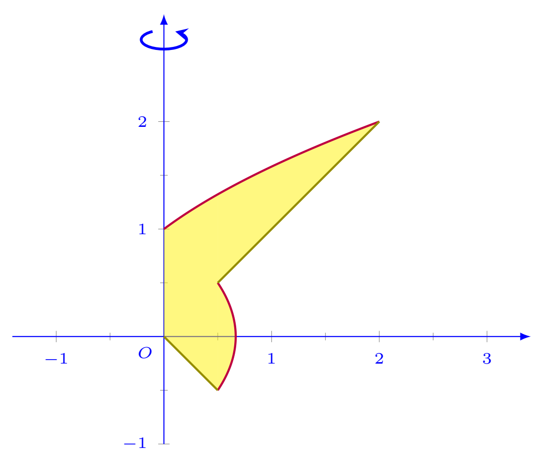
Hence the volume of the solid obtained by revolving the green area is

$$\int_{\frac{1}{2}}^{2}(2\pi x)\left(\sqrt{\frac{3}{2}x+1}-x\right)dx$$

By repeating the above process to the other parts, we obtain that the total volume of the solid is

$$\begin{aligned}V &= \int_{-\frac{2}{3}}^{-\frac{1}{2}}(-2\pi x)\cdot 2\sqrt{\frac{3}{2}x+1}dx + \int_{-\frac{1}{2}}^{0}(-2\pi x)\left(\frac{1}{2}-x\right)dx \\ &\quad + \int_{0}^{\frac{1}{2}}(2\pi x)\left(\sqrt{\frac{3}{2}x+1}-\frac{1}{2}\right)dx + \int_{\frac{1}{2}}^{2}(2\pi x)\left(\sqrt{\frac{3}{2}x+1}-x\right)dx \\ &= \frac{17}{135}\pi + \frac{5}{24}\pi + \left(\frac{121}{1080}+\frac{7\sqrt{7}}{270}\right)\pi + \left(\frac{749}{540}-\frac{7\sqrt{7}}{270}\right)\pi \\ &= \frac{11}{6}\pi\end{aligned}$$

Remark: by symmetry with respect to the y-axis, we could have considered the revolution around the y-axis of the following region only situated on the right side.



2. Write a proof for the following theorem

   Theorem Let $\{a_n\}_{n=0}^{\infty}$ be a sequence.
   - IF $\{a_n\}_{n=0}^{\infty}$ is eventually decreasing and not bounded below,
   - THEN $\{a_n\}_{n=0}^{\infty}$ is divergent to $-\infty$.

   Write a proof directly from the definitions. You will need to use the definitions of "eventually decreasing", "not bounded below", and "divergent to $-\infty$". As usual, pay attention to the proof structure.

Method 1:

This method relies on the following lemma, that we need to prove first (notice that it answers the question around 4:30 in Video 11.4).

Lemma: $\{a_n\}_{n=0}^{\infty}$ is bounded from below if and only if $\{a_n\}_{n=0}^{\infty}$ is eventually bounded from below.

Proof: Recall that $\{a_n\}_{n=0}^{\infty}$ is bounded from below means

$$\exists L \in \mathbb{R},\, \forall n \in \mathbb{N},\, a_n \geq L \tag{1}$$

and that $\{a_n\}_{n=0}^{\infty}$ is eventually bounded from below means

$$\exists n_0 \in \mathbb{N},\, \exists L \in \mathbb{R},\, \forall n \in \mathbb{N},\, (n > n_0 \implies a_n \geq L) \tag{2}$$

The non-trivial part consists in proving that if a sequence is eventually bounded from below then it is bounded from below.
Assume that $\{a_n\}_{n=0}^{\infty}$ is eventually bounded from below.

- By (2), there exist $n_0 \in \mathbb{N}$ and $L' \in \mathbb{R}$ such that if $n > n_0$ then $a_n \geq L'$.
- Set $L = \min(a_0, a_1, \dots, a_{n_0}, L')$.
- Let $n \in \mathbb{N}$.
- Either $n = 0, \cdots, n_0$ and then $a_n \geq L$ or $n > n_0$ and then $a_n \geq L' \geq L$.

We have proven that $\{a_n\}_{n=0}^{\infty}$ is bounded from below by L. Q.E.D.

We can now answer the question. We know that

- $\{a_n\}_{n=0}^{\infty}$ is eventually decreasing, i.e.

$$\exists n_0 \in \mathbb{N},\, \forall n, m \in \mathbb{N},\, (n > m \geq n_0 \implies a_n < a_m) \tag{3}$$

- $\{a_n\}_{n=0}^{\infty}$ is not bounded from below, by the above lemma, it means that $\{a_n\}_{n=0}^{\infty}$ satisfies the negation of (2), i.e.

$$\forall L \in \mathbb{R},\, \forall n_0 \in \mathbb{N},\, \exists n \in \mathbb{N},\, (n > n_0 \text{ and } a_n < L) \tag{4}$$

and we want to show that $\lim\limits_{n \to +\infty} a_n = -\infty$, i.e.

$$\forall M \in \mathbb{R},\, \exists n_0 \in \mathbb{N},\, \forall n \in \mathbb{N},\, (n > n_0 \implies a_n < M)$$

- Let $M \in \mathbb{R}$.
- By (3), there exists $n_1 \in \mathbb{N}$ such that

$$\forall n, m \in \mathbb{N},\, (n > m \geq n_1 \implies a_n < a_m) \tag{5}$$

- By (4) applied to M and n_1, there exists $n_0 \in \mathbb{N}$ such that

$$n_0 > n_1 \text{ and } a_{n_0} < M \tag{6}$$

- Let $n \in \mathbb{N}$, assume that $n > n_0$.
- Since $n > n_0 > n_1$, by (5) we know that $a_n < a_{n_0}$.
- Morever, by (6) $a_{n_0} < M$.
- Hence we have well that if $n > n_0$ then $a_n < M$.

Q.E.D.

Method 2:
We know that

- $\{a_n\}_{n=0}^{\infty}$ is eventually decreasing, i.e.

$$\exists n_0 \in \mathbb{N},\, \forall n, m \in \mathbb{N},\, (n > m \geq n_0 \implies a_n < a_m) \tag{7}$$

- $\{a_n\}_{n=0}^{\infty}$ is not bounded from below, i.e.

$$\forall L \in \mathbb{R},\, \exists n \in \mathbb{N},\, a_n < L \tag{8}$$

and we want to show that $\lim\limits_{n \to +\infty} a_n = -\infty$, i.e.

$$\forall M \in \mathbb{R},\, \exists n_0 \in \mathbb{N},\, \forall n \in \mathbb{N},\, (n > n_0 \implies a_n < M)$$

- Let $M \in \mathbb{R}$.

- By (7), there exists $n_1 \in \mathbb{R}$ such that $\forall n, m \in \mathbb{N}$,

$$n > m \geq n_1 \implies a_n < a_m \tag{9}$$

- Let $L = \min(a_0, a_1, \ldots, a_{n_1-1}, M)$.
- By (8), there exists $n_0 \in \mathbb{N}$ such that $a_{n_0} < L$.
- Notice that for $i = 0, 1, \ldots, n_1 - 1$, $a_i \geq L$. Hence, $n_0 \geq n_1$.
- Let $n \in \mathbb{N}$ and assume that $n > n_0$.
- Then, since $n > n_0 \geq n_1$, by (9), we have that $a_n < a_{n_0} < L \leq M$.
- Hence if $n > n_0$ then $a_n < M$.

Q.E.D.

3. In this problem we only consider sequences that are **positive and divergent to** ∞.

For each of the following statements, decide whether they are true or false. If true, prove it. If false, give a counterexample.

(a) IF $\{x_n\}_n$, $\{y_n\}_n$, $\{z_n\}_n$ are sequences such that $x_n << y_n$ and $y_n << z_n$ THEN $x_n << z_n$.

The statement is true. Below is a proof:
Assume that $x_n << y_n$ and $y_n << z_n$. We want to show that $x_n << z_n$.
Since $\frac{x_n}{z_n} = \frac{x_n}{y_n} \cdot \frac{y_n}{z_n}$, we have by the product law for limits

$$\lim_{n\to\infty} \frac{x_n}{z_n} = \lim_{n\to\infty} \frac{x_n}{y_n} \cdot \frac{y_n}{z_n} = 0 \cdot 0 = 0$$

Hence $x_n << z_n$. Q.E.D.

(b) For every sequence $\{x_n\}_n$, there exists a sequence $\{y_n\}_n$ such that $y_n << x_n$

The statement is true. Below is a proof:
Let $\{x_n\}_n$ be a positive sequence which diverges to $+\infty$.
We define a sequence $\{y_n\}_n$ by

$$y_n = \sqrt{x_n}$$

Notice that $y_n > 0$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} \sqrt{x_n} = +\infty$.

Then

$$\lim_{n\to\infty} \frac{y_n}{x_n} = \lim_{n\to\infty} \frac{1}{\sqrt{x_n}} = 0$$

Hence $y_n << x_n$. Q.E.D.

(c) IF $\{x_n\}_n$ and $\{y_n\}_n$ are sequences such that $x_n << y_n$
THEN there exists a sequence $\{z_n\}_n$ such that $x_n << z_n << y_n$.

The statement is true. Below is a proof:
Let $\{x_n\}_n$ and $\{y_n\}_n$ be two positive sequences which diverge to $+\infty$.
We define a sequence $\{z_n\}_n$ by

$$z_n = \sqrt{x_n y_n}$$

Notice that $z_n > 0$ and $\lim_{n\to\infty} z_n = \lim_{n\to\infty} \sqrt{x_n}\sqrt{y_n} = +\infty$.

Then, by continuity of the square root function at 0^+, we have

$$\lim_{n\to\infty} \frac{x_n}{z_n} = \lim_{n\to\infty} \sqrt{\frac{x_n}{y_n}} = 0$$

and

$$\lim_{n\to\infty} \frac{z_n}{y_n} = \lim_{n\to\infty} \sqrt{\frac{x_n}{y_n}} = 0$$

Hence we have well $x_n << z_n$ and $z_n << y_n$, as required. Q.E.D.

(d) For every sequence $\{x_n\}_n$, there exists a sequence $\{y_n\}_n$ such that for every $a > 0$, $(x_n)^a << y_n$

The statement is true. Below are two proofs:

Proof 1:

Let $\{x_n\}_n$ be a positive sequence which diverges to $+\infty$.
We define a sequence $\{y_n\}_n$ by

$$y_n = e^{x_n}$$

Notice that $y_n > 0$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} e^{x_n} = +\infty$.

Let $a > 0$. By studying the function $f(x) = e^{\frac{x}{a+1}} - \frac{x}{a+1}$, we can easily check that for $x > 0$,

$$e^{\frac{x}{a+1}} > 1 + \frac{x}{a+1} > \frac{x}{a+1}$$

Hence $e^x > \left(\frac{x}{a+1}\right)^{a+1} = Cx^{a+1}$ where $C = \frac{1}{(a+1)^{a+1}}$.
Therefore, we proved that

$$\forall x > 0,\ 0 < \frac{x^a}{e^x} < \frac{1}{Cx}$$

We derive from the above inequality that

$$\forall n \in \mathbb{N},\ 0 < \frac{(x_n)^a}{y_n} = \frac{(x_n)^a}{e^{x_n}} < \frac{1}{Cx_n}$$

Then, using the squeeze theorem, we obtain that

$$\lim_{n\to+\infty} \frac{(x_n)^a}{y_n} = 0$$

We have proven that for any $a > 0$ we have $(x_n)^a << y_n$, as required.

Q.E.D.

Proof 2:

Let $\{x_n\}_n$ be a positive sequence which diverges to $+\infty$.
We define a sequence $\{y_n\}_n$ by

$$y_n = (x_n)^n$$

Notice that $y_n > 0$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} e^{n\ln(x_n)} = +\infty$.
Let $a \in \mathbb{N}$, then

$$\lim_{n\to+\infty} \frac{(x_n)^a}{y_n} = \lim_{n\to+\infty} e^{(a-n)\ln(x_n)} = 0$$

We have proven that for any $a > 0$ we have $(x_n)^a << y_n$, as required.

Q.E.D.