MAT 137Y: Calculus!
Problem Set 7
Due on Thursday, January 31 by 11:59pm via crowdmark

1. Let $a, b, c, k \in \mathbb{R}$. Compute the following limit

$$\lim_{x\to 0} \frac{\int_{ax}^{bx}\left[\int_{ct}^{kt} e^{-s^2} ds\right] dt}{\cos x - 1}$$

Make sure you explain what you are doing and to justify the steps you take. You will not get any credit for a bunch of calculations without any words. If you find the calculation hard, it may be helpful to give names to some of the intermediate functions.

We first introduce intermediate functions that will be useful to explain the computation.

- The function $f(x) = e^{-x^2}$ is continuous on $\mathbb{R}$. Hence, according to FTC–1, the function

$$F(x) = \int_0^x e^{-s^2} ds$$

is well-defined and differentiable on $\mathbb{R}$ and moreover $F'(x) = e^{-x^2}$.

- The function

$$g(x) = \int_{cx}^{kx} e^{-s^2} ds = \int_0^{kx} e^{-s^2} ds - \int_0^{cx} e^{-s^2} ds = F(kx) - F(cx)$$

is differentiable on $\mathbb{R}$ since F is.

- Since g is continuous, as a differentiable function, we can apply FTC–1 again to show that the function

$$G(x) = \int_0^x g(t)dt$$

is well-defined, differentiable on $\mathbb{R}$ and that $G'(x) = g(x)$.

- Set

$$H(x) = \int_{ax}^{bx}\left[\int_{ct}^{kt} e^{-s^2} ds\right] dt = G(bx) - G(ax)$$

- We know that H is differentiable since G is.

Notice that

$$\frac{\int_{ax}^{bx}\left[\int_{ct}^{kt} e^{-s^2}ds\right]dt}{\cos x - 1} = \frac{H(x)}{\cos(x)-1}$$

We are going to compute the limit of the question by applying L'Hôpital's rule twice.

<u>Step 1</u>: computation of $\lim_{x\to 0} \frac{\frac{d}{dx}\big(H(x)\big)}{\frac{d}{dx}\big(\cos(x)-1\big)}$.

First notice that

$$\frac{\frac{d}{dx}\big(H(x)\big)}{\frac{d}{dx}\big(\cos(x)-1\big)} = \frac{bG'(bx)-aG'(ax)}{-\sin(x)} = \frac{ag(ax)-bg(bx)}{\sin(x)}$$

(a) Since g is continuous, we have

$$\lim_{x\to 0}\Big(ag(ax)-bg(bx)\Big) = ag(0)-bg(0) = 0-0 = 0$$

and since sin is continuous, we have

$$\lim_{x\to 0}\sin(x) = \sin(0) = 0$$

(b) Using the differentiation rules (including the chain rule), we know that the numerator is differentiable and that

$$\begin{aligned}\frac{d}{dx}\big(ag(ax)-bg(bx)\big) &= a^2g'(ax)-b^2g'(bx)\\ &= a^2ke^{-(kax)^2}-a^2ce^{-(cax)^2}-b^2ke^{-(kbx)^2}+b^2ce^{-(cbx)^2}\end{aligned}$$

(c) The denominator is differentiable and $\sin'(x)=\cos(x)$.

(d) There exists a small interval centered at 0 such that $\sin(x)\neq 0$ and $\sin'(x) = \cos(x)\neq 0$ on this interval, except at 0 for sin.

(e) By continuity of the involved functions and by the limit laws, we get

$$\begin{aligned}\lim_{x\to 0}\frac{a^2ke^{-(kax)^2}-a^2ce^{-(cax)^2}-b^2ke^{-(kbx)^2}+b^2ce^{-(cbx)^2}}{\cos(x)} &= \frac{a^2k-a^2c-b^2k+b^2c}{1}\\ &= (a^2-b^2)(k-c)\end{aligned}$$

Hence, according to L'Hôpital's rule,

$$\lim_{x\to 0}\frac{\frac{d}{dx}\big(H(x)\big)}{\frac{d}{dx}\big(\cos(x)-1\big)} = (a^2-b^2)(k-c)$$

Step 2: computation of $\lim_{x \to 0} \dfrac{H(x)}{\cos(x) - 1}$.

(a) By continuity of G and cos, we know that

$$\lim_{x \to 0} H(x) = G(0) - G(0) = 0$$

and that

$$\lim_{x \to 0} \cos(x) - 1 = \cos(0) - 1 = 0$$

(b) According to the chain rule, H is differentiable.

(c) The denominator is differentiable and $\frac{d}{dx}\big(\cos(x) - 1\big) = -\sin(x)$.

(d) There exists a small interval centered at 0 such that $\cos(x) - 1 \neq 0$ and $-\sin(x) \neq 0$ on this interval except at 0.

(e) We have proved in Step 1 that $\lim_{x \to 0} \dfrac{\frac{d}{dx}\big(H(x)\big)}{\frac{d}{dx}\big(\cos(x) - 1\big)} = (a^2 - b^2)(k - c)$

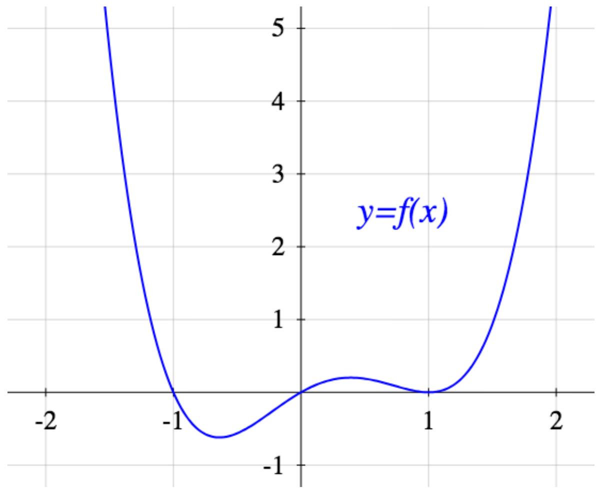
Hence, according to L'Hôpital's rule:

$$\lim_{x \to 0} \frac{H(x)}{\cos(x) - 1} = (a^2 - b^2)(k - c)$$

Conclusion:

$$\lim_{x \to 0} \frac{\int_{ax}^{bx} \left[\int_{ct}^{kt} e^{-s^2} ds\right] dt}{\cos x - 1} = (a^2 - b^2)(k - c)$$

2. Below is the graph of the function f:



The domain of f is $\mathbb{R}$ and the graph continues to the right and to the left as you expect. We define a new function H by

$$H(x) = \int_0^{\int_0^x f(t)dt} f(s)ds$$

How many local maxima and local minima does H have? Give the approximate x-coordinate for each one of them.

Hint: If you are having trouble computing the derivative of H, we recommend again that you give names to the intermediate functions.

- We know that f is continuous on $\mathbb{R}$ hence, according to FTC–1,

$$F(x) = \int_0^x f(t)dt$$

  is well-defined, differentiable on $\mathbb{R}$ and moreover $F' = f$.

- Notice that

$$H(x) = \int_0^{\int_0^x f(t)dt} f(s)ds = \int_0^{F(x)} f(s)ds = F(F(x))$$

- Hence, by the chain rule, H is differentiable on $\mathbb{R}$ and

$$H'(x) = F'(x)F'(F(x)) = f(x)f(F(x))$$

- We know that the local extrema are reached at the critical points, i.e. at $x \in \mathbb{R}$ such that $H'(x) = 0$.

Since
$$H'(x) = 0 \Leftrightarrow \Big(f(x) = 0 \text{ or } f(F(x)) = 0\Big)$$
we are going to study these two cases separately.

<u>Case 1:</u> $f(x) = 0$
According to the graph, there are three solutions which are -1, 0, and 1.

<u>Case 2:</u> $f(F(x)) = 0$
Using the previous case, we are looking for $x \in \mathbb{R}$ such that $F(x) = -1$, 0 or 1.

The graph of f allows us to write the following variation table:

x	$-\infty$		-1		0		1		$+\infty$
$F'(x) = f(x)$		$+$	0	$-$	0	$+$	0	$+$	
$F(x)$	$-\infty$	↗	α	↘	0	↗	β	↗	$+\infty$

Moreover, still using the graph of f, we know that

- $\alpha = F(-1) = \int_0^{-1} f(t)dt = -\int_{-1}^{0} f(t)dt \in (0,1)$ (that's the yellow area below), i.e.
$$0 < \alpha < 1$$
- $\beta = F(1) = \int_0^1 f(t)dt \in (0,1)$ (that's the first part of the blue area below), i.e.
$$0 < \beta < 1$$

Since F is continuous, as a differentiable function, we can use the IVT and the monotonicity of F to conclude that:

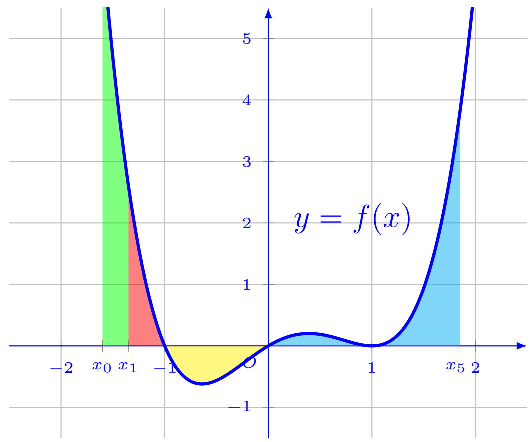
(a) There is exactly one $x_0 \in (-\infty, -1)$ such that $F(x_0) = -1$.

(b) There is exactly one $x_1 \in (-\infty, -1)$ such that $F(x_1) = 0$.

(c) $F(0) = 0$

(d) There is exactly one $x_5 \in (1, +\infty)$ such that $F(x_5) = 1$.

We can find approximate values for these x_i by looking at the graph of f:



(a) We know that x_1 is the point such that the red area is equal to the yellow one, so that $F(x_1) = 0$, we read graphically $x_1 \simeq -1.35$.

(b) We know that x_0 is the point such that the green area is equal to 1, so that $F(x_0) = -1$, we read graphically $x_0 \simeq -1.60$.

(c) We know that x_5 is the point such that the blue area is equal to 1, so that $F(x_5) = 1$, we read graphically $x_5 \simeq 1.85$.

Therefore the critical points are:
$x_0 \simeq -1.60$, $x_1 = -1.35$, $x_2 = -1$, $x_3 = 0$, $x_4 = 1$ and $x_5 \simeq 1.85$.

We still need to figure out if they are local extrema or not.

- Notice that

$$0 < x < y \Rightarrow 0 < F(x) < F(y) \Rightarrow 0 < H(x) < H(y)$$

  Hence H is strictly increasing on $(0, +\infty)$.
  So H has no local extrema at x_4 and x_5.

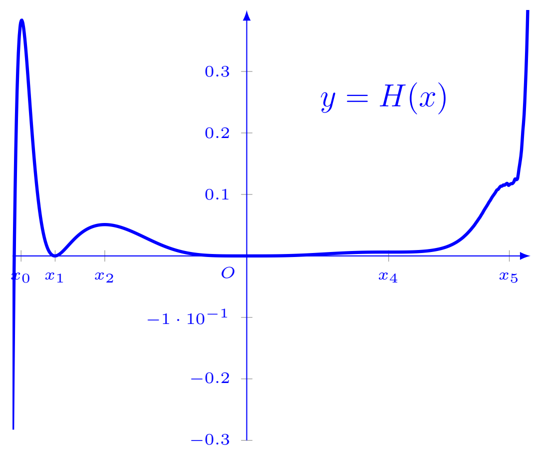
- We know that $H(x_1) = H(x_3) = 0$ whereas H is positive around these points, so they are local min.

- Since f is continuous on $[x_1, x_3]$, we know by the EVT that f has a min and a max on this interval. The max is among the endpoints and the critical points x_2. Since $H(x_1) = H(x_3) = 0$ and $H(x_2) = F(F(x_2)) = F(1) > 0$, we deduce that $H(x_2)$ is a local max.

- Since f is continuous on $[-10, x_1]$, we know by the EVT that f has a min and a max on this interval.
  The max is among the endpoints and the critical point x_0.
  Since $H(-10) < 0$, $H(x_1) = 0$ and $H(x_0) = F(F(x_0)) = F(-1) = \alpha > 0$, we deduce that $H(x_0)$ is a local max.

Conclusion :

- There is a local max at $x_0 \simeq -1.60$.
- There is a local min at $x_1 \simeq -1.35$.
- There is a local max at $x_2 = -1$.
- There is a local min at $x_3 = 0$.



3. In this problem, you are going to compute the exact value of the integral $I = \displaystyle\int_{-2}^{1} \left(x^2+1\right)\, dx$ using Riemann sums. Let us call $f(x) = x^2 + 1$. Since f is continuous on $[-2, 1]$, we know it is integrable. Hence, its value can be computed using Riemann sums as video 7.11 explains.

For every natural number n, let us call P_n the partition that splits $[-2, 1]$ into n *equal* sub-intervals. Notice that $\lim_{n\to\infty} ||P_n|| = 0$. Hence, we can write $I = \lim_{n\to\infty} S^\star_{P_n}(f)$ where $S^\star_{P_n}(f)$ is any Riemann sum for f and P_n. In particular, to make things simpler, we will use Riemann sums always choosing the right end-point to evaluate f on each subinterval.

(a) What is the length of each sub-interval in P_n?

The length of $[-2, 1]$ is $1 - (-2) = 3$.
Since P_n consists in breaking $[-2, 1]$ into n subintervals of same length, we obtain that the length of each subinterval is $\frac{3}{n}$.

(b) Let us write $P_n = \{x_0, x_1, \ldots, x_n\}$. Find a formula for x_i in terms of i and n.

$$x_i = -2 + i\frac{3}{n} = -2 + \frac{3i}{n}$$

(c) Since we are using the right-endpoint, it means we are picking $x_i^\star = x_i$. Use your above answers to obtain an expression for $S^\star_{P_n}(f)$ in the form of a sum with sigma notation.

$$\begin{aligned}
S^\star_{P_n}(f) &= \sum_{i=1}^{n} \Big((x_i - x_{i-1}) f(x_i^\star) \Big) \\
&= \sum_{i=1}^{n} \Big((x_i - x_{i-1}) f(x_i) \Big) \\
&= \sum_{i=1}^{n} \left(\frac{3}{n} f\left(-2 + \frac{3i}{n}\right) \right) \\
&= \frac{3}{n} \sum_{i=1}^{n} \left(\left(-2 + \frac{3i}{n}\right)^2 + 1 \right)
\end{aligned}$$

(d) Using the formulas

$$\sum_{i=1}^{N} i = \frac{N(N+1)}{2}, \quad \sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}, \quad \sum_{i=1}^{N} i^3 = \frac{N^2(N+1)^2}{4}$$

if needed, add up the expression you got to obtain a nice, compact formula for $S^{\star}_{P_n}(f)$ without any sums or sigma symbols.

$$\begin{aligned}
S^{\star}_{P_n}(f) &= \frac{3}{n}\sum_{i=1}^{n}\left(\left(-2+\frac{3i}{n}\right)^2+1\right) \\
&= \frac{3}{n}\sum_{i=1}^{n}\left(\frac{9}{n^2}i^2-\frac{12}{n}i+5\right) \\
&= \frac{27}{n^3}\sum_{i=1}^{n} i^2 - \frac{36}{n^2}\sum_{i=1}^{n} i + \frac{15}{n}\sum_{i=1}^{n} 1 \\
&= \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{36}{n^2}\cdot\frac{n(n+1)}{2} + \frac{15}{n}n \\
&= \frac{27n(n+1)(2n+1) - 108n^2(n+1) + 90n^3}{6n^3} \\
&= \frac{36n^3 - 27n^2 + 27n}{6n^3} \\
&= \frac{12n^3 - 9n^2 + 9n}{2n^3}
\end{aligned}$$

(e) Calculate $\lim_{n\to\infty} S^{\star}_{P_n}(f)$. This number will be the exact value of $\int_{-2}^{1}(x^2+1)dx$.

Method 1: with the simplified form.

$$\begin{aligned}
\lim_{n\to\infty} S^{\star}_{P_n}(f) &= \lim_{n\to\infty}\frac{12n^3-9n^2+9n}{2n^3} \\
&= \frac{12}{2} \\
&= 6
\end{aligned}$$

Method 2: without the simplified form.

$$\begin{aligned}
\lim_{n\to\infty} S^{\star}_{P_n}(f) &= \lim_{n\to\infty}\left(\frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{36}{n^2}\cdot\frac{n(n+1)}{2} + 15\right) \\
&= \frac{27\cdot 2}{6} - \frac{36}{2} + 15 \\
&= 9 - 18 + 15 \\
&= 6
\end{aligned}$$

Hence

$$\int_{-2}^{1}(x^2+1)dx = 6$$

(f) (**Do not submit.**) Now repeat all the previous steps using left endpoints instead of right endpoints. You should get the exact same final answer.

$$\begin{aligned}
S^{\star}_{P_n}(f) &= \sum_{i=1}^{n}\Big((x_i - x_{i-1})f(x_i^{\star})\Big) \\
&= \sum_{i=1}^{n}\Big((x_i - x_{i-1})f(x_{i-1})\Big) \\
&= \sum_{i=1}^{n}\left(\frac{3}{n}f\left(-2+\frac{3(i-1)}{n}\right)\right) \\
&= \frac{3}{n}\sum_{i=1}^{n}\left(\left(-2+\frac{3(i-1)}{n}\right)^2+1\right) \\
&= \frac{3}{n}\sum_{i=1}^{n}\left(\frac{9i^2}{n^2}-\frac{18i}{n^2}-\frac{12i}{n}+\frac{9}{n^2}+\frac{12}{n}+5\right) \\
&= \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}-\frac{54}{n^3}\cdot\frac{n(n+1)}{2}-\frac{36}{n^2}\cdot\frac{n(n+1)}{2}+\frac{27}{n^3}n+\frac{36}{n^2}n+\frac{15}{n}\cdot n
\end{aligned}$$

Hence

$$\lim_{n\to\infty} S^{\star}_{P_n}(f) = \frac{27\cdot 2}{6} - 0 - \frac{36}{2} + 0 + 0 + 15 = 6$$

(g) (**Do not submit.**) Verify that your answer is correct using antiderivatives and FTC 2.

$$\int_{-2}^{1}(x^2+1)dx = \left[\frac{x^3}{3}+x\right]_{-2}^{1} = \frac{1}{3}+1+\frac{8}{3}+2 = 6$$