MAT 137Y: Calculus!
Problem Set 6
Due on MONDAY, January 21 by 11:59pm via crowdmark

1. Let f be a function with domain $\mathbb{R}$. Assume f is decreasing and bounded below. Let A be the infimum of f. Prove that

$$\lim_{x \to \infty} f(x) = A$$

Notes: You will need to use the definition of infimum of a function and the definition of $\lim_{x \to \infty} f(x)$. Do not make any unwarranted assumptions about the function f; for example, do not assume that f is continuous or that $\lim_{x \to \infty} f(x)$ exists.

<u>Method 1:</u> with the ε-characterization of the infimum.
We know that:

(a) f is decreasing, i.e.

$$\forall x, y \in \mathbb{R},\ x < y \Rightarrow f(x) > f(y) \tag{1}$$

(b) $A = \inf\{f(x),\ x \in \mathbb{R}\}$, i.e.

$$\begin{cases} \forall x \in \mathbb{R},\ A \leq f(x) & (2) \\ \forall \varepsilon > 0,\ \exists x \in \mathbb{R},\ f(x) < A + \varepsilon & (3) \end{cases}$$

We want to show that $\lim_{x \to +\infty} f(x) = A$, i.e.

$$\forall \varepsilon > 0,\ \exists M \in \mathbb{R},\ \forall x \in \mathbb{R},\ \big(x > M \Rightarrow |f(x) - A| < \varepsilon\big)$$

- Let $\varepsilon > 0$.
- By (3), there exists $M \in \mathbb{R}$ such that $f(M) < A + \varepsilon$.
- Let $x \in \mathbb{R}$. Assume that $x > M$.
- Then, by (1), $f(x) < f(M) < A + \varepsilon$. Hence $f(x) - A < \varepsilon$.
- By (2), $f(x) - A \geq 0$, so $|f(x) - A| = f(x) - A < \varepsilon$.
- Therefore we have well that if $x > M$ then $|f(x) - A| < \varepsilon$.

Q.E.D.

Method 2: without the ε-characterization of the infimum.
We know that:

(a) f is decreasing, i.e.

$$\forall x, y \in \mathbb{R},\ x < y \Rightarrow f(x) > f(y) \tag{4}$$

(b) $A = \inf\{f(x),\ x \in \mathbb{R}\}$, which means that

- A is a lower bound of f, i.e.

$$\forall x \in \mathbb{R},\ A \leq f(x) \tag{5}$$

- and, it is the greatest one, i.e.

$$\text{if } B \text{ is a lower bound of } f \text{ then } A \geq B \tag{6}$$

Actually, we are going to use the contrapositive of 6:

$$\text{If } A < B \text{ then } B \text{ isn't a lower bound of } f \tag{6'}$$

We want to show that $\lim_{x \to +\infty} f(x) = A$, i.e.

$$\forall \varepsilon > 0,\ \exists M \in \mathbb{R},\ \forall x \in \mathbb{R},\ \big(x > M \Rightarrow |f(x) - A| < \varepsilon\big)$$

- Let $\varepsilon > 0$.
- Since $A + \varepsilon > A$, we derive from (6') that $A + \varepsilon$ isn't a lower bound of f.
- Hence there exists $M \in \mathbb{R}$ such that $f(M) < A + \varepsilon$.
- Let $x \in \mathbb{R}$. Assume that $x > M$.
- Then, by (4), $f(x) < f(M) < A + \varepsilon$. Hence $f(x) - A < \varepsilon$.
- By (5), $f(x) - A \geq 0$, so $|f(x) - A| = f(x) - A < \varepsilon$.
- Therefore we have well that if $x > M$ then $|f(x) - A| < \varepsilon$.

Q.E.D.

2. Consider the set

$$\begin{aligned} B &= \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\} \\ &= \left\{x \in \mathbb{R} \mid \exists n \in \mathbb{Z} \text{ s.t. } n > 0 \text{ and } x = \frac{1}{n}\right\} \end{aligned}$$

I define the function g by the equation

$$g(x) = \begin{cases} 0 & \text{if } x \in B \\ 1 & \text{if } x \notin B \end{cases}$$

In this question you are going to study the integrability of the function g on $[0, 1]$. Before you begin, sketch the graph of this function to understand it better. As always, make sure to justify all your answers.

(a) What is the upper integral $\overline{I_0^1}(g)$?

Let $P = \{0 = x_0 < x_1 < \cdots < x_N = 1\}$ be a partition of $[0, 1]$.
Then the upper Darboux sum of g with respect to P is

$$U_P(g) = \sum_{k=1}^{N} \left((x_k - x_{k-1}) \sup_{[x_{k-1}, x_k]} g \right)$$

For each k, we know there exists $a \in [x_{k-1}, x_k]$ which is not a rational number.
Particularly $a \notin B$ and $g(a) = 1$.
Moreover, $\forall x \in [x_{k-1}, x_k]$, $g(x) \leq 1$.
Hence $\sup_{[x_{k-1}, x_k]} g = 1$.
Therefore

$$U_P(g) = \sum_{k=1}^{N} \left((x_k - x_{k-1}) \sup_{[x_{k-1}, x_k]} g \right) = \sum_{k=1}^{N} (x_k - x_{k-1}) = x_N - x_0 = 1 - 0 = 1$$

Thus $U_P(g) = 1$ for any partition P of $[0, 1]$.
Finally, $\overline{I_0^1}(g) = 1$ as the infimum of the set containing only 1.

Q.E.D.

(b) Prove the following claim:

"*For every positive integer n, and for every $\varepsilon > 0$, there exists a partition P of $[0,1]$ such that $L_P(g) > 1 - \frac{1}{n} - \varepsilon$.*"

Let $n \in \mathbb{N}_{>0}$. Let $\varepsilon > 0$.

If $n = 1$, then, for $P = \{0 < 1\}$, $L_P(g) = \inf_{[0,1]} g = 0 > -\varepsilon$.
Hence, we may assume that $n > 1$.

Set $\alpha = \frac{\min\left(\frac{\varepsilon}{2(n-1)}, \frac{1}{2n(n-1)}\right)}{2}$, then

$$\alpha > 0 \tag{7}$$

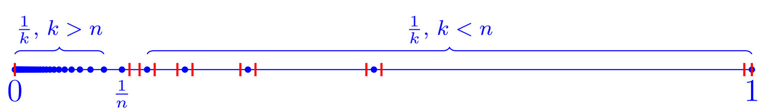
$$\alpha < \frac{\varepsilon}{2(n-1)} \tag{8}$$

$$\alpha < \frac{1}{2n(n-1)} \tag{9}$$

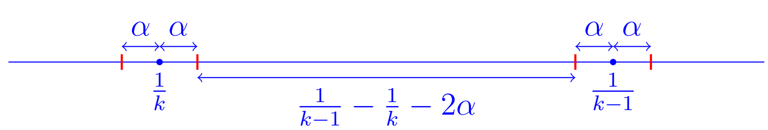
We introduce the partition

$$P = \left\{ 0 < \frac{1}{n} + \alpha < \frac{1}{n-1} - \alpha < \frac{1}{n-1} + \alpha < \cdots < \frac{1}{k} - \alpha < \frac{1}{k} + \alpha < \cdots < \frac{1}{2} - \alpha < \frac{1}{2} + \alpha < 1 - \alpha < 1 \right\}$$

The two following figures may help you to vizualize this partition.
First this is what $B \cap [0,1]$ looks like (the boundaries of the subintervals of the partition are in red):



Next, this is a zoom around two consecutive elements of $B \cap [0,1]$ where $k \leq n$:



(*The function vanishes on the subintervals containing an element of B, represented by a $\bullet$ on the figures. Then, the idea consists in choosing α small enough so that the subintervals without an element of B are the big enough: the bigger they are, the bigger is the lower Darboux sum in order to answer the question.*)

The partition is well-constructed (i.e. the inequalities are satisfied):

- For $1 < k < n$, we derive from (7) that we have well

$$\frac{1}{k} - \alpha < \frac{1}{k} + \alpha$$

- For $1 < k \leq n$, we derive from (9) that we have well

$$\frac{1}{k} + \alpha < \frac{1}{k-1} - \alpha$$

(*You can see it graphically: the smallest possible distance is for $k = n$ and then $\frac{1}{n} + \alpha < \frac{1}{n-1} - \alpha$ by (9). Actually, that's how you can deduce the condition (9).*
Then you can also check it algebraically:

$$2\alpha < \frac{1}{n(n-1)} \leq \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}.$$

)

In order to compute $L_P(g)$, notice that:

- Since $g\left(\frac{1}{n}\right) = 0$, we have

$$\inf_{\left[0, \frac{1}{n}+\alpha\right]} g = 0$$

- Since $g(1) = 0$, we have

$$\inf_{[1-\alpha, 1]} g = 0$$

- For $1 < k < n$, we derive from $g\left(\frac{1}{k}\right) = 0$ that

$$\inf_{\left[\frac{1}{k}-\alpha, \frac{1}{k}+\alpha\right]} g = 0$$

- For $1 < k \leq n$, we derive from $\left[\frac{1}{k} + \alpha, \frac{1}{k-1} - \alpha\right] \cap B = \varnothing$ that

$$\inf_{\left[\frac{1}{k}+\alpha, \frac{1}{k-1}-\alpha\right]} g = 1$$

(*It means that the only non-zero summands in the lower Darboux sum are the ones coming from the intervals without an element of B, as expected! See the above figures.*)

Hence

$$\begin{aligned}
L_P(g) &= \sum_{k=2}^{n} \left(\left(\frac{1}{k-1} - \alpha \right) - \left(\frac{1}{k} + \alpha \right) \right) \cdot 1 \\
&= \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k} - 2\alpha \right) \\
&= \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k} \right) - 2\alpha(n-1) \\
&= 1 - \frac{1}{n} - 2\alpha(n-1)
\end{aligned}$$

Then, we derive from (8) that

$$L_P(g) > 1 - \frac{1}{n} - \varepsilon$$

Q.E.D.

(c) Prove the following claim:

"*For every* $\varepsilon > 0$, *there exists a partition* P *of* $[0,1]$ *such that* $L_P(g) > 1 - \varepsilon$."

Hint: Use your answer to Question 2b. Mind your quantifiers and your proof structure.

Let $\varepsilon > 0$.

Set $n = \left\lfloor \frac{2}{\varepsilon} \right\rfloor + 1$ so that $-\frac{1}{n} > -\frac{\varepsilon}{2}$.

Applying Question 2.(b). to n and $\frac{\varepsilon}{2}$, we deduce that there exists a partition P such that

$$L_P(g) > 1 - \frac{1}{n} - \frac{\varepsilon}{2} > 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = 1 - \varepsilon$$

Q.E.D.

(d) What is the lower integral $\underline{I_0^1}(g)$?

We are going to prove that $\underline{I_0^1}(g) = 1$.
Recall that $\underline{I_0^1}(g) = \sup\{L_P(g),\ P \text{ is a partition of } [0,1]\}$.

Method 1: using the ε-characterization of the supremum.

- Let P be a partition of $[0,1]$.
  Then, according to Question 2. (a), $L_P(g) \leq U_P(g) = 1$.
- Let $\varepsilon > 0$.
  Then, according to Question 2. (c)., there exists a partition P such that $L_P(g) > 1 - \varepsilon$.

We have shown that

$$\begin{cases} \text{For any partition } P \text{ of } [0,1],\ L_P(g) \leq 1 \\ \text{For any } \varepsilon > 0, \text{ there exists a partition } P \text{ of } [0,1] \text{ such that, } 1 - \varepsilon < L_P(g) \end{cases}$$

Hence $\underline{I_0^1}(g) = \sup\{L_P(g),\ P \text{ is a partition of } [0,1]\} = 1$. Q.E.D.

Method 2: without the ε-characterization of the supremum.
We have to check the following two parts of the definition of the supremum.

- 1 is an upper bound of the $L_P(g)$:
  Let P be a partition of $[0,1]$.
  Then, according to Question 2. (a), $L_P(g) \leq U_P(g) = 1$.
- If T is an upper bound of the $L_P(g)$ then $1 \leq T$:
  Actually, we are going to show the contrapositive:

  if $T < 1$ then T isn't an upper bound.

  Let $T < 1$. Set $\varepsilon = 1 - T > 0$.
  Then, according to Question 2. (c)., there exists a partition P such that $L_P(g) > 1 - \varepsilon = T$.
  Hence T isn't an upper bound of the $L_P(g)$.

Hence $\underline{I_0^1}(g) = \sup\{L_P(g),\ P \text{ is a partition of } [0,1]\} = 1$. Q.E.D.

(e) **[Do not submit]** Is g integrable on $[0, 1]$?

According to Question 2. (a). and Question 2. (d)., we have

$$\underline{I_0^1}(g) = \overline{I_0^1}(g) = 1$$

Hence g is integrable on $[0, 1]$ and

$$\int_0^1 g(x)dx = 1$$

Q.E.D.